Area maximizing surfaces in Lorentzian spaces

Magdalena Caballero[a], José A. S. Pelegrín[b] and Rafael M. Rubio[a]

[a] Departamento de Matemáticas, Facultad de Ciencias,
Universidad de Córdoba, 14071 Córdoba, Spain,
E-mail: magdalena.caballero@uco.es, rmrubio@uco.es

[b] Departamento de Matemática Aplicada y Estadística,
Universidad CEU San Pablo, 28003 Madrid, Spain
E-mail: jose.sanchezpelegrin@ceu.es

Abstract

In this paper we provide new results for area maximizing compact spacelike surfaces with boundary embedded in Lorentz-Minkowski space, as well as establish the uniqueness of the Dirichlet problem for maximal graphs in the aforementioned space. Moreover, we extend our results to more general Lorentzian spaces that admit an infinitesimal timelike symmetry.

MSC 2010: 53C42, 53C50, 58J32.

Keywords: Maximal surface, Area maximizing surface, Lorentzian space.

1 Introduction

Historically, minimal surface theory in Riemannian Geometry arises to answer the problem of characterizing those surfaces which have the smallest area (area minimizing) among all surfaces with the same boundary [23]. Recall that in variational terms, minimal surfaces are defined as critical points of the area functional for compactly supported normal variations, which is equivalent to the surface having zero mean curvature. Nevertheless, minimal surfaces are not in general area minimizing [21].

An analogous problem in Lorentzian geometry is given in terms of the so-called maximal surfaces. An isometrically embedded surface Σ in a Lorentzian space, i.e., a three-dimensional Lorentzian manifold (M, g), is called spacelike if the induced metric on Σ is Riemannian. Thus, an embedded spacelike surface in (M, g) is said to be maximal if its mean curvature function vanishes identically. From a variational point of view, maximal surfaces are also critical points of the area functional for compactly supported normal variations and they have been deeply studied due to their importance in General Relativity (see [4], [7] and [19], for instance). Unfortunately, not all maximal surfaces are area maximizing in any arbitrary Lorentzian space. Therefore, the first step towards the characterization of area maximizing spacelike surfaces is the study of stable maximal surfaces (see [7], [9], [10], [13] and [14]). A stable maximal surface is area maximizing relative to nearby spacelike surfaces with the same boundary.

Returning to the Riemannian case, in [3] the authors prove that if the area of the image by the Gauss map of a domain of a minimal surface in Euclidean space is smaller than 2π, then the minimal surface defined on the closure of the domain is stable. Hence, it is natural to ask if it is possible to obtain an analogous result in Lorentz-Minkowski space $\mathbb{L}^3$. However, in Section 2 we show that every compact maximal surface with boundary in Lorentz-Minkowski space is stable, highlighting againg the great differences that appear between Riemannian and Lorentzian Geometry in this type of problems [25].

Therefore, once the stability of compact maximal surfaces with boundary in $\mathbb{L}^3$ is ensured, the next natural step is to study wether they maximize the area. Indeed, in Section 3 we prove under some natural assumptions that every compact maximal surface with boundary in $\mathbb{L}^3$ is area maximizing, see Theorem 6. Moreover, this surface is unique. As a consequence of this result we can establish the uniqueness of the Dirichlet problem for maximal surfaces in Lorentz-Minkowski space (Theorem 8).

In Section 4 we extend our results to more general Lorentzian spaces that admit an infinitesimal timelike symmetry. In particular, we will focus on standard stationary spacetimes since, as it can be seen in [9, Thm. C] and [10, Thm. 8], there exist some Lorentzian spacetimes where maximal surfaces are not even stable and, therefore, they do not maximize the area. Finally, we will devote Section 5 to prove the uniqueness of the Dirichlet problem for maximal surfaces in standard static spacetimes.

We would like to highlight that our techniques can also be used to prove the area maximizing character of maximal hypersurfaces in standard stationary spacetimes of arbitrary dimension. However, in this article we will just focus on the classical problem for area maximizing surfaces in three-dimensional Lorentzian spaces.

2 Preliminaries

Let $\mathbb{L}^3$ be the three-dimensional Lorentz-Minkowki space, i.e., the differentiable manifold $\mathbb{R}^3 = \{(t,x,y) : t,x,y \in \mathbb{R}\}$ endowed with the Lorentzian metric

$$g = -dt^2 + dx^2 + dy^2. \tag{1}$$

Given $u : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ a smooth function, where Ω is an open domain with $\overline{\Omega}$ a compact set in $\mathbb{R}^2$, the graph

$$\Sigma_u = \{(u(x,y),x,y) : (x,y) \in \overline{\Omega}\}$$

constitutes a differentiable surface (with boundary) in $\mathbb{L}^3$ (in fact, a regular surface [11]). If we denote by g_0 the usual flat Riemannian metric of $\mathbb{R}^2$, the graph Σ_u is spacelike if and only if the induced metric $g_u = -du^2 + g_0$ on Ω via the graph is Riemannian, i.e., the function u satisfies

$$|Du| < 1,$$

being Du the gradient of u respect to the metric g_0. From now on, we will assume that the graph Σ_u is spacelike. It is easy to see that

$$N = \frac{1}{\sqrt{1-|Du|^2}}(\partial_t + Du)$$

is the unitary normal vector field in the same time orientation of the coordinate timelike vector field $\partial_t := \frac{\partial}{\partial t}$, i.e., $g(\partial_t, N) \leq -1$. Thus, the area of the graph is given by

$$\mathrm{Area}(\Sigma_u) = \int_\Omega \sqrt{1-|Du|^2}\; dx \wedge dy.$$

Let us consider a one-parameter family of spacelike graphs $\Sigma_{u+s\xi}$, where s is in an open interval containing 0 and ξ is a smooth function with $\xi|_{\partial\Omega} = 0$. Then, the first variation of the area functional for the spacelike graph defined by u is given by

$$\frac{d}{ds}\bigg|_{s=0} \mathrm{Area}(\Sigma_{u+s\xi}) = -\int_\Omega \frac{g_0(Du, D\xi)}{\sqrt{1-|Du|^2}} = \int_\Omega \xi \,\mathrm{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right). \tag{2}$$

As a direct consequence, the graph Σ_u is a stationary point for the area functional if u satisfies the following equation in divergence form

$$\mathrm{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0. \tag{3}$$

This is a nonlinear partial differential equation, which is elliptic thanks to the spacelike condition $|Du| < 1$. This equation is known in the literature as the maximal surface equation in Lorentz-Minkowski space.

Consider now the more general case of a (connected) embedded spacelike surface $\psi : \Sigma \longrightarrow \mathbb{L}^3$ in $\mathbb{L}^3$. We also denote by g the Riemannian metric induced in Σ by the ambient one and identify $\psi(\Sigma) \equiv \Sigma$ as long as it does not generate confusion. In this setting, every smooth function with compact support $\phi \in C_0^\infty(\Sigma)$ induces a normal variation of the original immersion ψ given by $\psi_s(p) = \exp_{\psi(p)}(s\phi(p)N(p))$ for $p \in \Sigma$, being N the unitary normal vector field to Σ in the same time orientation than ∂_t. Since ϕ has compact support and $\mathrm{Im}(\psi_0)$ is spacelike, there exists $\epsilon > 0$ such that $\psi_s(\Sigma) \equiv \Sigma_s$ is spacelike for every $|s| < \epsilon$, enabling us to define the area functional $\mathrm{Area}(\Sigma_s)$ as

$$\mathrm{Area}(\Sigma_s) = \int_\Sigma d\Sigma_s,$$

where $d\Sigma_s$ is the Riemannian area element induced by ψ_s on Σ. As it is well-kown, the first variation of the area (see, for example, [17]) is given by

$$\left.\frac{d}{ds}\right|_{s=0} \text{Area}(\Sigma_s) = 2\int_M \phi H d\Sigma, \tag{4}$$

being $H := -\frac{1}{2}\text{trace}\,(A)$ the mean curvature function associated to N, where A denotes the the shape operator associated to N. From (4) we clearly see that maximal surfaces in $\mathbb{L}^3$, i.e., with $H \equiv 0$, are critical points of the area functional for compactly supported normal variations. Moreover, the stability of this variational problem depends on the second variation of the area functional for maximal surfaces in $\mathbb{L}^3$, which is (see [7, Thm. 2.1])

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \text{Area}(\Sigma_s) = \int_\Sigma \left[\phi\Delta\phi - \text{trace}(A^2)\phi^2\right] d\Sigma, \tag{5}$$

where Δ denotes the Laplacian with respect to g. Therefore, we can define the associated quadratic form

$$Q(\phi,\phi) = \int_\Sigma \left[\Delta\phi - \text{trace}(A^2)\phi\right] \phi \; d\Sigma, \quad \phi \in C_0^\infty(\Sigma).$$

Analogously to the Riemannian case for minimal surfaces (see, for instance, [12] and [21]), stability of maximal surfaces in $\mathbb{L}^3$ is detected by the sign of the first eigenvalue of the stability operator, called Jacobi operator L, given by

$$Lu = \Delta u - \text{trace}(A^2)u. \tag{6}$$

Hence, we denote by $\lambda_1^L(\Sigma)$ its first eigenvalue on Σ, which is defined as

$$\lambda_1^L(\Sigma) = \inf_\Omega \lambda_1^L(\Omega), \tag{7}$$

where Ω is any bounded domain in Σ. The first eigenvalue $\lambda_1^L(\Omega)$ can be also given by the following variational characterization,

$$\lambda_1^L(\Omega) = \inf_{\phi\in C_0^\infty(\Omega),\ \phi\neq 0} \frac{\int_\Omega \left(|\nabla\phi|^2 + \text{trace}(A^2)\phi^2\right)}{\int_\Omega \phi^2}. \tag{8}$$

Thus, a maximal surface Σ is stable if and only if

$$\lambda_1^L(\Sigma) \geq 0. \tag{9}$$

Since $\text{trace}(A^2) \geq 0$, we easily deduce from (8) that every maximal surface in $\mathbb{L}^3$ is stable. Therefore, a natural question arises: *When is a maximal surface in* $\mathbb{L}^3$ *area maximizing?*

2.1 Spacelike graphs and spacelike surfaces

In this section we study, under certain natural geometrical conditions, the relation between spacelike surfaces in general and spacelike graphs, both of them with compact closure in $\mathbb{L}^3$. We will obtain that the elements of a wide class of spacelike surfaces are graphs. First, we obtain the next topological result which is fundamental for our goal.

Lemma 1 *Let X, Z be two topological spaces and assume that X is Hausdorff and compact. Then every surjective local homeomorphism $\pi : X \longrightarrow Z$ is a covering map. Moreover, the subset $\pi^{-1}(z) \subset X$ is finite, for all $z \in Z$.*

Proof. Let $z \in Z$ be an arbitrary point. The subset $\pi^{-1}(z)$ is closed and therefore compact. Moreover, there are no accumulation points in $\pi^{-1}(z)$. Otherwise, assume that $\bar{x}$ is an accumulation point. Since $\pi^{-1}(z)$ is compact we know that $\bar{x} \in \pi^{-1}(z)$. As a direct consequence, every open set $\Theta \subset X$ with $\bar{x} \in \Theta$ must contain infinite points of $\pi^{-1}(z)$, but this is contradictory, since π is a local homeomorphism.

Now, taking into account that X is Hausdorff, we can isolate each point of $\pi^{-1}(z)$ in an open set homeomorphic via π with its image. On the other hand, since $\pi^{-1}(z)$ is compact and has no accumulation points, its cardinal must be finite. □

We also need the following technical result.

Lemma 2 *Consider a connected embedded spacelike surface Σ in $\mathbb{L}^3$ whose topological closure is compact and such that its boundary $\partial\Sigma$ coincides with the boundary $\partial\Sigma_u$ of a spacelike graph Σ_u defined on a connected open domain $\Omega \subset \mathbb{R}^2$ whose topolological closure is simply connected. Then, the spacelike surface Σ is necessarily a graph on Ω and, consequently, it is contained in the cylinder $\mathbb{R} \times \Omega$*

Proof. Let $\pi : \mathbb{L}^3 \longrightarrow \mathbb{R}^2$ be the canonical projection $\pi(t, x, y) = (x, y)$. The restriction $\pi : \Sigma \longrightarrow \mathbb{R}^2$ is a smooth map whose differential increases the norm of the tangent vectors. As a direct consequence, π is a local diffeomorphism. Thus, $\tilde{\Omega} := \pi(\Sigma)$ is an open subset in $\mathbb{R}^2$.

Since $\overline{\Sigma}$ is compact and $\overline{\Omega}$ is simply connected, we have $\overline{\Omega} \subseteq \overline{\tilde{\Omega}}$. Let $p \in \partial\tilde{\Omega}$ and $q \in \overline{\Sigma}$ be such that $\pi(q) = p$. Suppose that $q \notin \partial\Sigma$. Then, there exists an open neighborhood $\mathcal{U}$ of q and therefore $\pi(\mathcal{U})$ is an open neighborhood of p, which is absurd. Thus, taking into account that Ω is connected and $\overline{\tilde{\Omega}}$ is compact, we can ensure that $\overline{\Omega} = \overline{\tilde{\Omega}}$. Furthermore, Lemma 1 ensures that $\pi : \overline{\Sigma} \longrightarrow \overline{\Omega}$ is a covering map, but taking into account that $\overline{\Omega}$ is simply connected, π must be a global diffeomorphism. □

From Lemma 2 we can obtain the following elegant corollary.

Corollary 3 *Consider a connected embedded spacelike surface Σ in $\mathbb{L}^3$ with compact topological closure whose boundary $\partial\Sigma$ satisfies that $\pi(\partial\Sigma)$ is a Jordan curve, being $\pi : \mathbb{L}^3 \longrightarrow \mathbb{R}^2$ the canonical projection $\pi(t, x, y) = (x, y)$. Let $\Omega \subset \mathbb{R}^2$ be the interior region bounded by $\pi(\partial\Sigma)$. Then, Σ is a graph on Ω and, consequently, it is contained in the cylinder $\mathbb{R} \times \Omega$*

3 Main results in Lorentz-Minkowski space

In this section we answer our initial question about area maximizing spacelike surfaces in Lorentz-Minkowki space $\mathbb{L}^3$. Our initial results will be for spacelike graphs and using Lemma 2 we will obtain a general result for embedded spacelike surfaces in $\mathbb{L}^3$.

Consider a connected open set $\Omega \subset \mathbb{R}^2$ such that $\overline{\Omega}$ is compact and let $\Sigma_u = \{(u(x,y), x, y) : (x,y) \in \Omega\}$ be a maximal graph. Note that we can extend its unitary normal future pointing vector field N and (as a direct consequence) its orthogonal space along the integral lines of ∂_t. Using this extended vector field we can define in the cylinder $\mathbb{R} \times \Omega$ the differential 2-form

$$\eta(X,Y) = \det(N, X, Y), \tag{10}$$

where $X, Y \in \mathfrak{X}(\mathbb{R} \times \Omega)$. Notice that if X, Y are orthogonal and unitary spacelike vector fields, then

$$|\eta(X,Y)| \geq 1, \tag{11}$$

and equality holds if and only if X and Y are orthogonal to the extended vector field N. Also, observe that $\eta|_{\Sigma_u}$ is the canonical Riemannian volume form on the spacelike graph. Moreover, if we denote by ζ the canonical Lorentzian volume element of $\mathbb{L}^3$, we have

$$d\eta = d(i_{_N}\zeta) = \overline{\mathrm{div}}(N)\zeta = 0, \tag{12}$$

since Σ_u is maximal, and where $\overline{\mathrm{div}}$ denotes the divergence operator in $\mathbb{L}^3$. Therefore, η is a Lorentzian calibration that calibrates Σ_u. This allows us to prove our first result for area maximizing graphs.

Theorem 4 *Let $\Omega \subset \mathbb{R}^2$ be a relatively compact domain with simply connected topological closure and let $u : \Omega \longrightarrow \mathbb{R}$ be a smooth function defining a maximal graph Σ_u in $\mathbb{L}^3$. Then, Σ_u is area maximizing among every spacelike surface Σ with compact closure such that $\partial\Sigma = \partial\Sigma_u$.*

Proof. From Lemma 2 we have that Σ and Σ_u are contained in the cylinder $\mathbb{R} \times \Omega$. Taking into account Σ_u and Σ are homologous, if we denote by $U \subset \mathbb{R} \times \Omega$ the connected subset of this cylinder such that $\partial U = \Sigma_u \cup \Sigma$, since η is closed from (12), we can use Stokes' theorem to obtain

$$\int_{\Sigma_u} \eta - \int_{\Sigma} \eta = \int_U d\eta = 0. \tag{13}$$

As a consequence, we have from (11) and (13)

$$\mathrm{Area}(\Sigma_u) = \int_{\Sigma_u} \eta = \int_{\Sigma} \eta \geq \mathrm{Area}(\Sigma). \tag{14}$$

□

Moreover, this maximal graph is unique as we see in the following result.

Theorem 5 *Let $\Omega \subset \mathbb{R}^2$ be a relatively compact domain with simply connected topological closure in $\mathbb{R}^2$ and let Σ_u and Σ_v be two maximal graphs over Ω in $\mathbb{L}^3$ with $\partial\Sigma_u = \partial\Sigma_v$. Then, $\Sigma_u = \Sigma_v$.*

Proof. From (14) we have $\text{Area}(\Sigma_u) \geq \text{Area}(\Sigma_v)$. If there is a point p in Ω where $T_{(v(p),p)}\Sigma_v$ is not orthogonal to $N(v(p), p)$, the inequality in (14) must be strict. However, this is absurd since Σ_v is also maximal. Taking into account that $\partial\Sigma_u = \partial\Sigma_v$ we conclude that $\Sigma_u = \Sigma_v$. □

Combining Theorems 4 and 5 and taking into account Lemma 2 we can enunciate the following result.

Theorem 6 *Let Σ be a connected embedded maximal surface in $\mathbb{L}^3$ with compact closure and whose boundary satisfies $\partial\Sigma \subset \mathbb{R} \times \partial\Omega$, where $\Omega = \pi(\Sigma) \subset \mathbb{R}^2$ is a connected open set with $\overline{\Omega}$ simply connected. Then, Σ is the unique area maximizing surface among every spacelike surface $\tilde{\Sigma}$ with compact closure such that $\partial\tilde{\Sigma} = \partial\Sigma$.*

Remark 7 Note that in Lorentz-Minkowski space, Plateau's problem does not make sense for causal curves (i.e., those whose velocity vector is always timelike or lightlike) due to the fact that there are no closed causal curves in $\mathbb{L}^3$. Thus, the natural Dirichlet problem in the Lorentzian case is in terms of spacelike graphs.

We would also like to emphasize that in Theorem 6 we only need the projection of the surface $\pi(\Sigma)$ to have simply connected topological closure, contrary to what happens in the classical Riemannian Plateau's problem, where the bounded domain must to be convex in order to ensure that the resulting minimal surface is area minimizing [20].

Moreover, dropping our assumptions about the connectedness of the surface and the simply connectedness of $\overline{\Omega}$ in Theorem 6 we can find the next counterexample: Consider the upper part of the spacelike Lorentzian catenoid in $\mathbb{L}^3$ whose equation is $\sinh^2 t = x^2 + y^2$, with $(x, y) \neq (0, 0)$ and $t > 0$, which is a graph over the plane punctured at the origin. Let us consider the maximal surface Σ_1 given by the part of the Lorentzian catenoid's graph over the annulus in $\mathbb{R}^2$ bounded by the circles of radii 1 and 2 centered at the origin. Denoting by Σ_2 the maximal surface which is the union of the disk of radius 1 centered at the origin in $\{\sinh^{-1}(1)\} \times \mathbb{R}^2$ and the disk of radius 2 centered at the origin in $\{\sinh^{-1}(2)\} \times \mathbb{R}^2$ we see that Σ_1 and Σ_2 have the same boundary but the area of Σ_2 is greater than the area of Σ_1.

These results allow us to study the uniqueness of the solutions to the Dirichlet problem for maximal graphs in $\mathbb{L}^3$. It is well-known that the existence of solutions to this Dirichlet problem is guaranteed (see [5] and [18, Chap. 12]). Indeed, in [5] the authors proved the uniqueness of this Dirichlet problem for locally Lipschitz functions by means of a different technique than the one used here, where as a consequence of Theorems 4 and 5 we can obtain the following result.

Theorem 8 *Let $\Omega \subset \mathbb{R}^2$ be a relatively compact domain with simply connected topological closure in $\mathbb{R}^2$ and φ a smooth function defined on $\partial\Omega$. Then, there exists a unique solution to the Dirichlet problem*

$$\begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0 & in\ \Omega \\ |Du| < 1 & in\ \Omega \\ u = \varphi & on\ \partial\Omega. \end{cases}$$

Moreover, this solution maximizes area among every spacelike graph in Ω *with the same boundary data.*

4 Area maximizing surfaces in standard stationary spacetimes

The $(n \geq 3)$-dimensional Lorentz-Minkowki space is a especially symmetrical element of a class of physically relevant Lorentzian manifolds called stationary spacetimes (see [27] and [26] for details). Recall that a stationary spacetime is a time-oriented Lorentzian manifold (M, g) which admits a complete timelike Killing vector field globally defined, $K \in \mathfrak{X}(M)$ (see [22] and [26]). Therefore, the vector field K satisfies $g(K, K) < 0$ and $\mathcal{L}_K g = 0$, where $\mathcal{L}$ denotes the Lie derivative. Note that K determines a time-orientation on M and recall that the stages ϕ_s of the global flow $\phi : M \times \mathbb{R} \longrightarrow M$ of K are isometries on M. Locally, a spacetime with a timelike Killing vector field K can be written as a standard stationary spacetime with respect to K, i.e., a product manifold $M = \mathbb{R} \times S$, endowed with the metric tensor

$$g = -\alpha^2 dt^2 + \omega \otimes dt + dt \otimes \omega + h, \tag{15}$$

where (S, h) is a Riemannian manifold, α is a positive smooth function on S and ω is a 1-form on S.

The properties of maximal hypersurfaces in spacetimes admitting a timelike Killing vector field have been previously studied in [1] [2], [8], [24] and [28], for instance. Nevertheless, in this section we will study the area maximizing properties of maximal surfaces in standard stationary spacetimes. The relevance of standard stationary spacetimes comes from the fact that under weak causality assumptions any spacetime admitting a complete timelike Killing vector field can be globally expressed as a standard stationary one.

Let us recall that a spacetime is chronological (resp. causal) if it does not contain any closed timelike (resp. causal) curve (see [29]). Going one step higher in the causal ladder, a spacetime is said to be distinguishing if for any points, $p \neq q$ implies both, $I^+(p) \neq I^+(q)$ and $I^-(p) \neq I^-(q)$, where $I^+(p)$ (resp. $I^-(p)$) denotes the chronological future (resp. past) of the point p (see [6] for details). In [15] it was shown that a chronological spacetime (M, g) with a global complete timelike conformal (or in particular, Killing) vector field K defined on it, admits a topological and differentiable global splitting $M = \mathbb{R} \times Q$, where Q represents the space of integral curves of K endowed with a natural manifold structure. Later, in [16] the authors characterized when such a splitting is extensible to the metric level. Indeed, a spacetime (M, g) admitting a complete timelike conformal Killing vector field admits a global standard conformastationary splitting if and only if (M, g) is distinguishing [16, Thm. 1.2]. In particular, if the spacetime (M, g) is distinguishing and admits a complete timelike Killing vector field, it can be expressed as a product manifold $M = \mathbb{R} \times S$, endowed with the metric (15).

Thus, let us consider a standard stationary spacetime $(M = \mathbb{R} \times S, g)$, being g the Lorentzian metric (15). Note that we have endowed (M, g) with the time orientation of the complete timelike Killing vector field $\partial_t := \frac{\partial}{\partial t}$. If we denote by (t, p) an arbitrary point in M, the projection $\pi : M \longrightarrow S$ is a smooth fuction. Note that the diffential of the function π restricted to any spacelike embedded hypersurface in M increases the norm of the tangent vectors. Consequently, a similar reasoning to the one in Lemma 2 holds in this case.

Thanks to [10, Cor. 7], we know that a compact embedded maximal surface with boundary Σ in a standard stationary space (M, g) is stable. Indeed, we can go one step further thanks to the following result.

Theorem 9 *Let Σ be a connected compact embedded maximal surface with boundary in a 3-dimensional standard stationary spacetime $(M = \mathbb{R} \times S, g)$ such that $\partial\Sigma \subset \mathbb{R} \times \partial\Omega$, where $\Omega = \pi(\Sigma) \subset S$ is a connected open set with $\overline{\Omega}$ simply connected. Then, Σ is the unique area maximizing surface among every compact embedded spacelike surface $\tilde{\Sigma}$ with $\partial\tilde{\Sigma} = \partial\Sigma$.*

Proof. Reasoning as in Lemma 2 and taking into account Lemma 1, we can see that Σ must be a spacelike graph $\Sigma = \{(u(p), p) : p \in \overline{\Omega}\}$, where $\overline{\Omega}$ is a simply connected compact domain in S and u is a smooth function on $\overline{\Omega}$ and the same holds for $\tilde{\Sigma}$, for a suitable function $\tilde{u}$ defined on $\overline{\Omega}$. Therefore, both surfaces are contained in the cylinder $\mathcal{C} = \mathbb{R} \times \overline{\Omega}$ and $\partial\Sigma \subset \mathbb{R} \times \partial\overline{\Omega}$. Observe that the generating lines of the cylinder are the integral curves of the complete timelike Killing vector field ∂_t.

Let N be the unitary timelike normal vector field on $\mathrm{Int}(\Sigma) := \Sigma \setminus \partial\Sigma$, which we can choose in the same time-orientation of ∂_t, i.e., $\overline{g}(N, \partial_t) < 0$. We can extend N to the interior of $\mathcal{C}$ via the flow ϕ of ∂_t. So, for each $p \in \mathrm{Int}(\Sigma)$, we define $\widetilde{N}(\phi(s, p)) = d\phi_s(N(p))$. Analogously, thanks to the isometries defined by the stages of the global flow, we can guarantee that there is a maximal surface $\Sigma_{(s,p)}$ whose unitary normal vector field at $\phi(s, p) \in \Sigma_{(s,p)}$ is $\widetilde{N}$. As a consequence, we obtain that

$$\overline{\mathrm{div}}(\widetilde{N}) = 0, \tag{16}$$

where $\overline{\mathrm{div}}$ denotes the divergence operator in the Lorentzian spacetime (M, g). Thus, we can define a differential 2-form η as in (10), which satisfies that $\eta|_\Sigma$ is the canonical Riemannian volume form on the spacelike graph Σ and $|\eta(X, Y)| \geq 1$ for any two unitary orthogonal spacelike vector fields $X, Y \in \mathfrak{X}(\mathcal{C})$, with equality holding if and only if X and Y are orthogonal to the extended vector field $\widetilde{N}$. Moreover, if ζ is the canonical Lorentzian volume element of M, from (16) we obtain

$$d\eta = \overline{\mathrm{div}}(\widetilde{N})\zeta = 0. \tag{17}$$

Therefore, η is closed. If we consider another compact embedded spacelike surface $\tilde{\Sigma}$ with $\partial\tilde{\Sigma} = \partial\Sigma$, reasoning as above we see that it satisfies $\tilde{\Sigma} \subset \mathcal{C}$. Hence, denoting by $U \subset \mathcal{C}$ the connected subset of this cylinder such that $\partial U = \Sigma \cup \tilde{\Sigma}$ we can use Stokes' theorem like in Theorem 4 to obtain

$$\int_{\Sigma} \eta - \int_{\tilde{\Sigma}} \eta = \int_{U} d\eta = 0. \tag{18}$$

Thus,

$$\text{Area}(\Sigma) = \int_{\Sigma} \eta = \int_{\tilde{\Sigma}} \eta \geq \text{Area}(\tilde{\Sigma}). \tag{19}$$

We prove the uniqueness by assuming that $\tilde{\Sigma}$ is also maximal and reasoning as in the proof of Theorem 5 to conclude that $\Sigma = \tilde{\Sigma}$. □

5 Uniqueness of the Dirichlet Problem in standard static spacetimes

An interesting subclass of stationary Lorentzian spacetimes is given by the so called standard static spacetimes, which admit an orthogonal splitting. Given $(\mathbb{B}^2, g_{\mathbb{B}})$ a two-dimensional Riemannian manifold, the standard static spacetime $\mathbb{R}\,{}_h\!\times \mathbb{B}^2$ is the product manifold $\mathbb{R} \times \mathbb{B}^2$ endowed with the Lorentzian metric

$$\overline{g} = -h^2 \pi_{\mathbb{R}}^*(dt)^2 + \pi_{\mathbb{B}}^*(g_{\mathbb{B}}), \tag{20}$$

where $h \in C^\infty(\mathbb{B})$ is a positive function called warping function and $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{B}}$ are the canonical projections of $\mathbb{R} \times \mathbb{B}^2$ onto $\mathbb{R}$ and $\mathbb{B}$ (see [22] for details). It is well known that the coordinate vector field $\partial_t := \frac{\partial}{\partial t}$ is a complete timelike Killing vector field [22, Chap. 12].

Consider a smooth function $u : \Omega \subset \mathbb{B}^2 \longrightarrow \mathbb{R}$, where Ω is an open domain with $\overline{\Omega}$ a compact set in $\mathbb{B}^2$. The graph

$$\Sigma_u = \{(u(x,y), x, y) : (x,y) \in \overline{\Omega}\}$$

constitutes a smooth compact embedded surface (with boundary) in $\mathbb{R}\,{}_h\!\times \mathbb{B}^2$. We can denote by g_u the Riemannian metric induced on Ω via the graph Σ_u, which is given by

$$g_u = -h^2 du^2 + g_{\mathbb{B}}.$$

This metric is Riemannian (i.e., the graph Σ_u is spacelike) if and only if u satisfies

$$|Du| < \frac{1}{h}, \tag{21}$$

where Du denote the gradient of u with respect to the metric $g_{\mathbb{B}}$. In this case, it is easy to see that

$$N = \frac{h}{\sqrt{1 - h^2 |Du|^2}} \left(\frac{1}{h^2} \partial_t + Du \right) \tag{22}$$

is the unitary timelike normal vector field to the graph in the same time orientation than ∂_t, i.e., $\overline{g}(\partial_t, N) \leq -1$. Thus, the area of the graph is given by

$$\mathrm{Area}(\Sigma_u) = \int_\Omega \sqrt{1 - h^2|Du|^2}\; d\mu,$$

where $d\mu$ is the canonical Riemannian volume element in $\mathbb{B}$. The spacelike graph Σ_u is a stationary point for the area functional (maximal case) if and only if u satisfies the following nonlinear PDE, which is elliptic thank to the spacelike condition (21).

$$\mathrm{div}\left(\frac{hDu}{\sqrt{1-h^2|Du|^2}}\right) = -\frac{g_{\mathbb{B}}(Du, Dh)}{\sqrt{1-h^2|Du|^2}}. \tag{23}$$

where div denotes the divergence operator in the Riemannian manifold $(\mathbb{B}^2, g_{\mathbb{B}})$. In the literature, equation (23) together with the constraint (21) is known as the maximal surface equation in a standard static space.

Taking into account Theorem 9 and the absolute maximizing character of a maximal surface with boundary among the spacelike surfaces with the same boundary, we obtain the following uniqueness result.

Theorem 10 *Let $\Omega \subset \mathbb{B}^2$ be an open set with simply connected compact closure and let φ be a smooth function defined on $\partial\Omega$. Suppose that u is a solution of the Dirichlet problem*

$$\begin{cases} \mathrm{div}\left(\frac{hDu}{\sqrt{1-h^2|Du|^2}}\right) = -\frac{g_{\mathbb{B}}(Du,Dh)}{\sqrt{1-h^2|Du|^2}} & \text{in } \Omega \\ |Du| < \frac{1}{h} & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{24}$$

Then, u is be unique. Moreover, this solution maximizes area among every spacelike graph in Ω with the same boundary data.

Example 11 As a direct application of Theorem 10, consider an arbitrary standard static space $\mathbb{R}_h\times \mathbb{B}^2$. It is easy to see that the spacelike surfaces given by the level sets $\{t = t_0, t_0 \in \mathbb{R}\}$ are a family of maximal surfaces in $\mathbb{R}_h\times \mathbb{B}^2$, which are called spacelike slices [2]. In fact, each spacelike slice is totally geodesic. If we consider $\Omega \subset \mathbb{B}^2$ an open set such that its closure $\overline{\Omega}$ is compact and φ a constant function $\varphi = u_0$, $u_0 \in \mathbb{R}$ on $\partial\Omega$, then the only solution to the Dirichlet problem (24) in this case is the constant graph $u = u_0$ on $\overline{\Omega}$, i.e., a piece of the corresponding spacelike slice.

Acknowledgements

This paper has been partially supported by Spanish MINECO and ERDF project MTM2016-78807-C2-1-P.

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