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Article

Optimization Problems in Spanish Differential Calculus Books Published in the 18th Century

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Abstract: History of mathematics and mathematics education research allows us to know, among other issues: the influence that certain textbooks have had on the teaching of school mathematics, in academic or professional training, during a certain historical period; as well as the scientific advances achieved in each historical period and their incorporation into the teaching of the subject matter. In this work, we focus our attention on the applications of the method of finding maxima and minima included in the textbooks published during the 18th century in Spain. Specifically, we identify the approach of the algorithm used, the shortcomings or deficiencies that the posing of the proposed problems may have, the verification of the nature of the optimal points obtained and the consideration—or not—of the negative solutions in the process of resolution.

Keywords: 18th century; textbooks; maxima and minima problems; differential calculus

MSC: 01A50; 97-03

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1. Introduction

The history of mathematics education is characterised by its interest in the multiple factors that have influenced the development of mathematical knowledge throughout history. On the one hand, it examines the curricula, teaching resources and legislative documents relating to mathematics education of each period, but also all the elements involved in the process of teaching and learning the discipline, such as the work or educational practice of textbook authors or educators and the social, political and academic events that influenced them [1].

In this sense, as Maz, Torralbo and Rico [2] note, research on the history of mathematics education shows and explains the process of incorporating new mathematical advances into the teaching of the subject and identifies how the social, political and academic context influences the way in which they are approached and disseminated within the current educational system.

Thus, in this field of study, in addition to those that address the state of the art [1] and the different research methodologies [3–5], there have been many studies that examine curricula, educational legislation, teaching resources and textbooks [6,7], identify the influence of philosophical, political, social, economic and cultural tendencies on mathematics education [8,9], analyse the evolution of the teaching role [10] and determine the contribution of historical character or educational institutions to mathematics education [11–14].

A review of the literature in Spain shows that the topics of greatest interest in this field involve the study of the implementation and development of curricula in institutions where the study of mathematics was cultivated [15,16], identifies the contributions of the relevant figures in the teaching of the discipline [17,18], or analyses the textbooks used for mathematics education in each period from different approaches [19–24].

It is in the latter line of research that the present work is framed. Since the appearance of the printing press and until a few decades ago, the textbook has been the main teaching resource [25], the most important source of information and dissemination of knowledge and the only support in educational institutions for teachers and students [26]. On the other hand, textbooks reflect all aspects of school knowledge of the time, as well as the didactical strategies that teachers used in their lessons [27].

This is the reason why the analysis of textbooks allows us to observe how knowledge was transmitted and applied in academic, political and social situations, as well as to show the relationship between scientific and academic knowledge [21]. In addition to providing us with pedagogical information, the analysis of textbooks let us know information about the development and evolution of mathematical concepts and methods, about the different views from which they have been approached and it provides us with the justification for their current form [28]. However, since books are written and used in a specific historical context and are addressed to a specific educational community, we can also see in them the reflection of habits and traditions related to the roles of all members of the educational community, from the student to the higher authorities [3].

There is a great deal of historical research on mathematics textbooks as the main source of documentation developed at the international level, and it deals with a variety of topics. Among them, the work carried out by Li and Li [13] is noteworthy, in which the characteristics of several series of textbooks are identified in order to determine the changes they have undergone following the educational reforms implemented in the 21st century. Another example is the research addressed by Ageron and Hedfib [29], in which a 16th century Arabic treatise on arithmetic is analysed, identifying the influences from the Spanish treatises of Marco Aurel and Juan de Ortega. Christianidis and Megremi [30] also examined how Diophantus' Arithmetica influenced mathematical texts written up to the early Middle Ages, and Friedman [31], in turn, explored 19th century mathematicians' conceptions of instrumental geometry, taking as a starting point the book *La geometria del compasso* published in 1797 by Lorenzo Mascheroni.

In a similar way, the analysis of old textbooks used for the teaching and learning of mathematics in Spain has generated a growing interest among researchers in the history of mathematics education. Thus, Puig and Fernández [32] present in their work some preliminary ideas to tackle the study of Marco Aurel's *Arithmetica Algebratica*, the first printed book on algebra written in Spanish; Gómez [33] describes the use, application and explanation of the rule of two as a method for solving arithmetic problems throughout history. Other authors, such as Muñoz-Escolano and Oller-Marcén [22], focus their attention on the prologues of algebraic texts published in Spanish during the 16th century. Likewise, Madrid, López-Esteban and Jiménez-Fanjul [34] identify the didactical strategies employed in three textbooks used to teach mathematics at the Academy of Midshipmen in Cadiz; Sánchez-Sierra and González-Astudillo [35] study the negative solutions of a geometric problem in analytical geometry textbooks used in Spain during the 19th century, and León-Mantero, Maz-Machado and Madrid [36] analyse Juan Cortázar's *Tratado de Álgebra Elemental* (Treatise on Elementary Algebra), one of the most relevant textbooks for the teaching of mathematics in secondary schools during the 19th century in Spain.

Despite all that has been researched and analysed on the identification of the bases of the current curriculum and the study of the origin of the problems that arise in the process of teaching and learning mathematics, there are still numerous research problems that remain unaddressed. For this reason, and under the "Dig where you stand" lemma, in this work, we analyse the exercises and problems proposed on the application of the method of maxima and minima included in the textbooks used in different educational institutions, in which the study of mathematics was cultivated during the 18th century, for the introduction and consolidation of the study of infinitesimal calculus in Spain.

2. Historical Context

Although there is evidence that infinitesimal calculus was studied and worked on in Spain at the beginning of the 18th century, it was not incorporated into the curricula of educational centres until the middle of the century. These were religious and military institutions dedicated to training engineers and professionals, such as the Royal Seminary of Nobles, the Imperial College, the College of Nobles of Cordelles, the Academy of Corps Guards and the Academy of Marine Guards of Cadiz [37,38].

Among the first people with knowledge of this branch of mathematics was Captain Pedro Padilla y Arcos, who published, between 1753 and 1756, what is considered to be the first textbook aimed at teaching the principles of infinitesimal calculus, the *Curso militar de Mathemáticas sobre las partes de estas ciencias pertenecientes al Arte de la Guerra, para el uso de la Real Academia establecida en el Quartel de Guardias de Corps* (Military Mathematics Course on the parts of these sciences pertaining to the Art of War, for the use of the Royal Academy established in the Corps Guards Barracks). Despite his efforts to incorporate the new methods of infinitesimal calculus into the teaching of mathematics at the time, his impact was limited. However, during the last quarter of the 18th century, with Spain's embrace of Enlightenment ideals, new institutions emerged with the aim of cultivating science, which made it necessary, on the one hand, to train teachers who could teach the new advances in mathematics achieved in Europe and, on the other, to design and publish textbooks for mathematical education that were in line with the teachings of the different institutions and that addressed all branches of mathematics, including infinitesimal calculus, which was making unstoppable progress in Europe [37,39]. Among them, the works of Benito Bails, Francisco de Villalpando, Pedro Giannini, Juan Justo García, Francisco Verdejo or Gabriel Císcar are considered of special relevance [39–46].

A review of the literature shows that there has been a great number of studies on the textbooks that served to introduce and consolidate this branch of mathematics in Spain during the 18th century [37,38,40–42,47,48]. However, a comparison of these from a mathematical or didactic point of view had not been addressed, with the exception of the work initiated by León-Mantero, Santiago and Gutiérrez-Arenas [49], which analyses the influences that Spanish textbooks received from those that served as a reference at the European level, with respect to the approach used, the contents included and their sequencing, the notation chosen and the method for calculating maxima and minima. This paper aims to complement the previous one by analysing the evolution of the method used to calculate maxima and minima of a function and how it was incorporated into the educational system through the exercises and problems proposed. First, we will identify the approach that characterises the algorithm used to solve these problems, and then compare examples presented in a similar way in several textbooks, analysing didactic aspects, such as the information given in the statement, the way of applying the necessary and sufficient conditions and the characteristics of the values obtained in the solution.

The results of this work bring benefits to the initial and in-service training of mathematics teachers, because on the one hand, they encourage mathematics to be seen as a human activity, they reflect the different approaches that have been made around mathematical concepts and structures, as well as the difficulties that arise when studying and working in the discipline [50], in particular, the obstacles that students often face when the method for calculating the maxima and minima of a function is addressed in the classroom.

Likewise, according to Schubring [51], when mathematics teachers are aware of the problems that have arisen throughout history and the solutions found, both in the scientific and scholastic development of the discipline, they are better able to face the difficulties that arise daily in their teaching work.

3. Materials and Methods

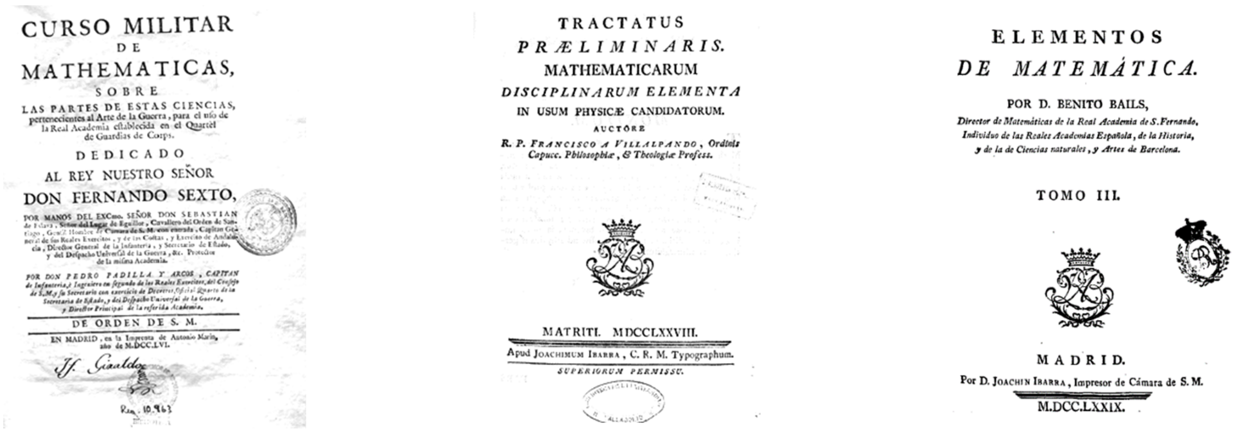
In addition to identifying the approach that permeates the method for calculating maxima and minima, this paper analyses the exercises and problems that the authors of textbooks used to introduce and consolidate the contents of infinitesimal calculus in eighteenth-century Spain. In particular, a comparison is made between examples with a similar approach and characteristics found in the actual statements of the examples and problems are reviewed, namely the resolution of the equation that imposes the necessary condition, in the consideration of negative solutions or in the application of the sufficient condition to check whether the value found is a maximum or a minimum.

This is a qualitative historical study that follows a descriptive and ex-post-facto methodology and uses the method of content analysis to examine and interpret the data, a well-established technique widely used in numerous research projects [21,36].

The paper by León-Mantero, Santiago and Gutiérrez-Arenas [49] establishes a list of the textbooks published in Spain during the 18th century that include differential and integral calculus contents and that were used to teach this branch of mathematics in different educational, military and religious institutions in the country (Figure 1). However, the work of Císcar [52] does not include applications of the method of maxima and minima, so that finally the list of textbooks analysed in this work is shown in Table 1, whose covers are shown in Figure 1.

The search for textbooks was carried out in virtual historical collections, such as the Cervantes Virtual Library, the Virtual Library of Andalusia or the Hispanic Digital Library of the National Library of Spain, as well as the digital repository of Google Books. Despite being aware of more authors with knowledge of infinitesimal calculus, their works were not included in this list and, therefore, were not analysed, because they are unpublished, manuscripts or cannot be located, which makes them unavailable for consultation.

The procedure for the analysis of the textbooks followed the recommendations given in Maz [53]: firstly, the statements and subsequent solutions of all the exercises and problems on the calculation of maxima and minima of functions present in the textbooks were defined as units of analysis, read, categorised and then analysed and compared.



(a) (b) (c)

(**d**) (**e**) (**f**)

Figure 1. Covers of analysed textbooks (**a**) Padilla and Arcos [54]; (**b**) Villalpando [55]; (**c**) Bails [56]; (**d**) Juan Justo García [57]; (**e**) Giannini [58]; and (**f**) Verdejo [59].

Table 1. Textbooks analysed and institutions to which they were directed [49].

Author	Title	Year of Publication	Educational Institution
Pedro Padilla y Arcos	*Curso militar de Mathemáticas sobre las partes de estas ciencias pertenecientes al Arte de la Guerra* (Tomo IV)	1756	Real Academia establecida en el Quartel de Guardias de Corps
Fernando Villalpando	*Tractatus Praeliminaris. Matehematicorum Disciplinarum Elementa in usum Physicae candidatorum*	1778	Noviciado de los capuchinos de Salamanca
Benito Bails	*Elementos de Matemática* (Tomo III)	1779	Academia de Bellas Artes de San Fernando
Pedro Giannini	*Curso Matemático para la enseñanza de los caballeros cadetes del Real Colegio Militar de Artillería* (Tomo III)	1795	Real Colegio de Caballeros Cadetes de Artillería de Segovia
Juan Justo García	*Elementos de Aritmética, Álgebra y Geometría*	1782	Universidad de Salamanca
Francisco Verdejo	*Compendio de Matemáticas puras y mixtas*	1802	Reales Estudios de San Isidro Real Casa de los Desamparados

4. Results

Before beginning to describe the results obtained in this work, it is necessary to point out that in the textual quotations included in this section, the original spelling, accentuation and punctuation that appear in the textbooks have been respected.

Next, we will identify the approach chosen by each of the authors to present the results and the steps of the algorithm of the method for calculating maxima and minima of functions, after which we will describe the differences found in the comparative analysis carried out on exercises and problems that are posed in a similar way.

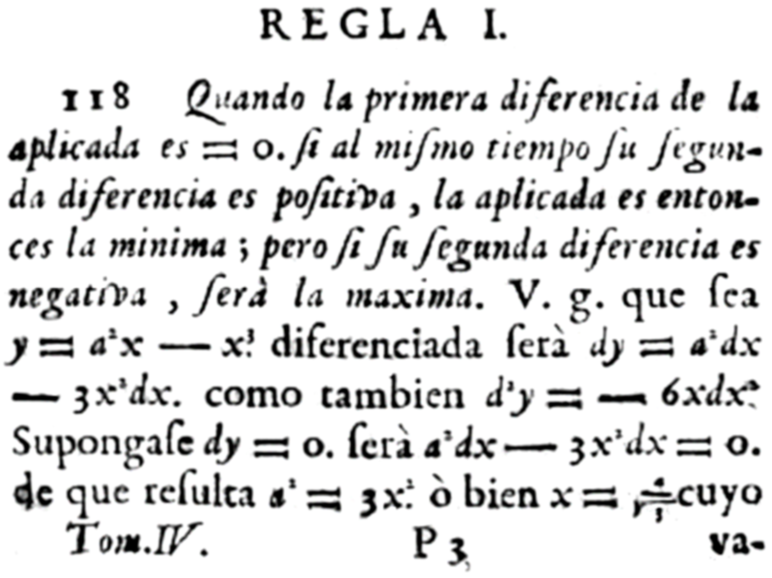
According to the research carried out by León-Mantero, Santiago and Gutiérrez-Arenas [49], in the textbooks published during the 18th century that include contents on infinitesimal calculus, we can find two differentiated approaches with respect to the method used to calculate maxima and minima of a function: the geometric approach and the analytical approach.

The main result that underlies the geometric approach is that if the quotient $\frac{dx}{dy}$ is equivalent to the value of the tangent of the angle that the curve forms with the OY axis (and similarly $\frac{dy}{dx}$ with respect to the OX axis), to find the values of the function in which the tangent lines are parallel to the axes, the necessary condition arises from equating the expression dx and dy to zero. In this case, to check that the values obtained (at $x = x_0$) are considered maximum or minimum, the authors indicate that the value obtained must be substituted in the curve, one value to its right and another to its left to check whether the ordinate in x_0 reaches greater or lesser values than in the other two.

Therefore, according to Giannini [58],

> Si la curva *AMN* tiene una ordenada *BM* máxima ó mínima, esta ordenada tendrá precisamente dos propiedades, es á saber, Iª. que la tangente en el punto *M* será paralela al exe ó diámetro *AG*, ó bien la misma ordenada *BM* será tangente á la curva en M: luego en el caso de ser la ordenada *BM* máxima ó mínima, ó será $dy = 0$, ó será $dx = 0$: 2ª. Que las ordenadas *bm*, *bm* infinitamente próximas á la ordenada *BM* serán ambas menores que la ordenada máxima *BM* intermedia. (pp. 248–249)
> (If the curve AMN has a maximum or minimum ordinate BM, this ordinate will have precisely two properties, namely, Iª that the tangent at the point M will be parallel to the axis or diameter AG, or else the same ordinate BM will be tangent to the curve at M: therefore in the case of the ordinate BM being maximum or minimum, it will either be $dy = 0$, or it will be $dx = 0$: 2. That the ordinates bm, bm infinitely close to the ordinate BM will both be less than the maximum intermediate BM ordinate).

However, in the analytical approach, which starts from the very definition of maximum or minimum and in which the aim is to calculate the values that are greater than other nearby points of the function, in the case of maximum, or smaller in the case of minimum, the condition imposed to carry out the calculations makes use of the Taylor series. Thus, Padilla y Arcos [54] indicates in the first place that: "Quando una Cantidad variable và creciendo hasta cierto punto que no puede ser mayor, y despues de èl decrece, se llama maxima" (p. 224) (When a variable quantity grows up to a certain point which cannot be greater, and after it stops growing, it is called a maximum). In a similar way, he defines minimum quantity. Then, he points out the property that underlies the necessary condition: "Si propuesta una Equacion se diferencia, è iguala à cero, la Equacion resultante contendrà las Cantidades máximas, ò mínimas de la propuesta" (p. 225) (If a proposed Equation is differentiated, or equals to zero, the resulting Equation will contain the maximum or minimum quantities of the proposed one). Finally, to check whether it is maximum or minimum, he checks the sign of the second difference (Figure 2).


REGLA I.

118 *Quando la primera diferencia de la aplicada es* $=$ o. *ſi al miſmo tiempo ſu ſegunda diferencia es poſitiva, la aplicada es entonces la minima; pero ſi ſu ſegunda diferencia es negativa, ſerà la maxima.* V. g. que ſea $y = a^2x - x^3$ diferenciada ſerà $dy = a^2dx - 3x^2dx$. como tambien $d^2y = -6xdx^2$. Supongaſe $dy =$ o. ſerà $a^2dx - 3x^2dx =$ o. de que reſulta $a^2 = 3x^2$. ò bien $x = \frac{a}{\sqrt{3}}$ cuyo

Tom. IV. P 3 va-

Figure 2. Rule I for testing whether a quantity is maximum or minimum. Source: Padilla and Arcos [54] (p. 225).

Thus, the text by Padilla y Arcos [54] uses the analytical approach, which is coherent considering that its main influence is McLaurin's Treatise of fluxions [60], in which the resolution algorithm is carried out through successive fluxions and Taylor's series development. On the other hand, according to the results obtained in León-Mantero, Santiago and Gutiérrez-Arenas [49], it is inferred that the works of Villalpando [55], Bails [56], Juan Justo García [57], Giannini [58] and Verdejo [59], were strongly influenced by the texts of the European authors, L'Hôpital [61], Euler [62], Bézout [63] or M. Cousin [64], and use the algorithm of calculating maxima and minima from a geometrical point of view.

Regarding the exercises and problems chosen by the authors to show the application of the method of maxima and minima, it is remarkable the great influence of the text by Bézout [63] from which Bails [56] extracts literally fourteen problems, and from which Verdejo (1802) [60], in turn, chooses six to include in his compendium. The author who includes the fewest applications is Villalpando [55], who only proposes two. Let us now study some of the characteristics found in the comparative analysis of the exercises and problems with respect to the formulation of the statement, the treatment given to the negative solutions, the type of points obtained after the application of the algorithm and the verification of the sufficient condition of maximum or minimum of the curve.

4.1. Shortcomings and Deficiencies in Problem Statements

One noteworthy aspect of some of the proposed examples is the redaction of the statements. Some of them lack part of the information that the reader needs to be able to solve them and the data that the author considered in the example can only be known after reading the proposed solution.

This is the case of the problems that aim to optimise the ordinates of the conics proposed by Bails [56], García [57] and Padilla and Arcos [55]. Each author considers a different conic, but only Bails [56] indicates in the statement the equation of the conic he wants to optimise. In this sense, Padilla y Arcos [54] only indicates in the statement that: "Se quiere hallar la maxima, ò minima aplicada en la Hyperbola Equilatera" (p. 231) (We want to find the maximum, or minimum applied in the Equilatera Hyperbola) and then in the resolution, he considers the equation of the hyperbola $y^2 = ax + x^2$. In the same way, García [58] proposes the exercise "Hallar la mayor ordenada y abscisa de la elipese" (p. 374) (Find the greatest ordinate and abscissa of the ellipse), indicating only the equation that will be taken into account in the resolution of the exercise, namely $aayy = 2abbx - bbxx$.

In relation to the previous statement, it is interesting to note the one proposed by Verdejo [60], which states: "Qüestion I. Hallar la mayor ordenada de una curva qualquiera KYX, cuyo exe es la recta KX, y el origen está en el punto X" (p. 144). (Question I. Find the greatest ordinate of any curve KYX, whose axis is the straight line KX, and the origin is at the point X). Although this statement is more general than the previous ones, the author solves this problem in the particular case where the curve is the circle of equation $y^2 = 2ax - x^2$ y $y^2 = aa - xx$, respectively (Figure 3).

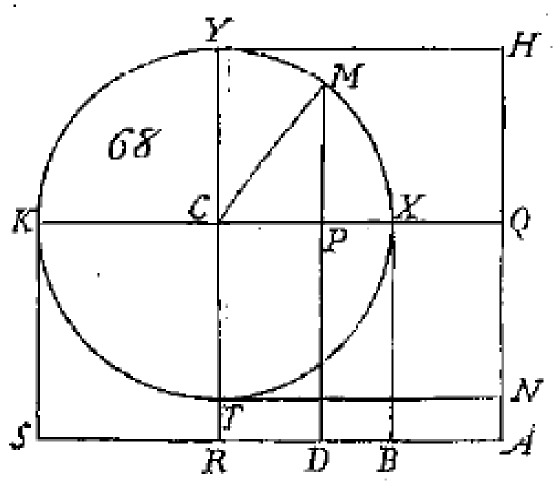


Figure 3. Graphic representation accompanying Qüestion I. Source: Verdejo [59] (plate IV).

Another example in which the data provided by the authors in the statement are not sufficient to solve the problem is the calculation of the dimensions of the parallelepiped with the smallest surface area that has a given volume. In order to obtain the three linear measurements of the parallelepiped (width–length–height), it is necessary for the statement to provide at least two pieces of data relating to the dimensions; however, only the volume of the body is indicated. Padilla y Arcos [54] states the problem using the following words. "Entre todos los Paralelepipedos de igual solidez se quiere hallar el que tenga menor superficie" (p. 235) (Among all the Paralelepipeds of equal solidity we want to find the one with the smallest surface). Giannini [58] chooses to propose: "Entre todos los paralelepípedos *FA* iguales á una cantidad dada, hallar él que tiene la mínima superficie convexâ" (p. 261) (Among all the *FA* parallelepipeds equal to a given quantity, find the one with the smallest convex surface).

In this case, the decisions taken by the authors in order to approach the resolution are undoubtedly dissimilar. Thus, Padilla y Arcos [54] chooses to indicate that the arbitrary side AB is taken as known without giving any explanation or reasons for this, and Giannini [58] reduces the problem on a first level to the search for parallelepipeds with the same height, shows that among those with the same height, the one with the smallest surface area is the one with a square base and, finally, calculates the one with the smallest surface area among those with a square base.

Another proposed problem that is striking for not being well defined in its statement is example 3, chosen by Padilla y Arcos [54], which says: "Se quiere dividir una recta *AB*. de suerte, que el Rectángulo hecho de las partes sea el mayor posible" (p. 232) (We want to divide a straight-line *AB*. so that the rectangle made up of its parts is the largest possible). This way of expressing the statement may lead the reader to think of dividing the length of the straight line into four parts, equal two by two, which constitute the four sides of the rectangle. However, in the resolution, the author divides the segment *AB* into only two parts and considers them as the base and the height of the rectangle.

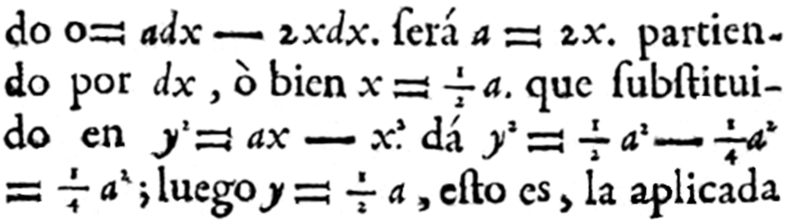
This problem does not appear again stated in this way in any of the textbooks; however, we can find in García [57] and Verdejo [59] a proposal whose objective is similar but lacks ambiguity. Thus, the example of García [57] says that "Se pide dividir un número a dado en dos partes tales, que el producto de la una por la otra, sea mayor que el de otras dos partes qualesquiera del mismo número" (p. 375) (We are asked to divide a given number a into two parts such that the product of one by the other is greater than that of any two other parts of the same number) and that of Verdejo asks "Dividir un número a en dos partes tales que el producto de una por la otra sea el mayor posible" (p. 149) (To divide a number a into two parts such that the product of one by the other is the greatest possible).

4.2. Consideration of Negative Values of Ordinates and Abscissae

Some of these examples, depending on the author and the context of the proposed example, take into account the negative solutions that appear when solving equations that are part of the solving algorithm, while others do not.

For example, in the exercise proposed by García [57], "Hallar la mayor ordenada y abscisa de la elipse" (p. 374) (Find the highest ordinate and abscissa of the ellipse), in which the following ellipse $aayy = 2abbx - bbxx$ is considered, it is observed that the possible maximum or minimum is reached at the point of abscissa $x = a$ after solving the equation obtained by equating the differentiated equation to zero. Substituting the value of the abscissa in the equation of the curve to obtain the value of the ordinate, the author obtains a second-degree equation whose solution is $y = \pm b$.

However, Padilla y Arcos [54] in solving the exercise, wrote: "Se quiere hallar la maxima, ò minima aplicada en el Circulo" (We want to find the maximum or minimum applied to the circle) and in which he takes into account the equation of the circumference $y^2 = ax - x^2$, that is to say, the one with centre at $(a/2{,}0)$ and radius $a/2$ does not consider the negative value of the ordinate, as shown in Figure 4.



do o $= adx - 2xdx$. ſerá $a = 2x$. partien-
do por dx, ò bien $x = \frac{1}{2}a$. que ſubſtitui-
do en $y^2 = ax - x^2$: dá $y^2 = \frac{1}{2}a^2 - \frac{1}{4}a^2$
$= \frac{1}{4}a^2$; luego $y = \frac{1}{2}a$, eſto es, la aplicada

Figure 4. Maximum and minimum problem in a circumference. Source: Padilla y Arcos [54].

With respect to the cases in which a curve is not being optimised, where in most cases, the aim is to calculate a linear measure, the authors are consistent with the nature of the variable they are calculating, i.e., they do not take into account the negative solutions of the equation that gives the value of x or y, although it would have been desirable for the authors to warn the reader of this fact by making some reference or indicating it with a note.

An example is the exercise proposed by Villalpando [55], who writes: "In omnibus triangulis rectangulis ejusdem areoe, illud invenire in quo altitude cathetorum AB+BC minor sit, quam fieri possit" (p. 286) (In all the right triangles with the same area, find the one which minimises the sum of the lengths of their catheti). Thus, the author considers that the area of the triangle is a; the legs AB and BC are $AB = x$ and $BC = 2a/x$; and the function to optimise, $x + 2a/x$. From there, he differentiates, equals zero dy and obtains the value of the ordinate $x = \sqrt{2a}$. It would not make sense to consider a solution of the abscissa with a negative sign.

4.3. Optimising Points on a Curve

Due to the theory underlying the algorithm used to calculate maxima and minima from the geometric approach, the values obtained do not correspond only to the maxima and minima of the ordinates or abscissae (Figure 5). Thus, according to García [57],

> 902. Quando se supone dx o $\frac{dx}{dy} = 0$ se supone cero ó nulo el ángulo que forma la curva ó su tangente con la ordenada: de suerte, que la tangente queda entonces paralela á las ordenadas como lo es la AR ó Br; y quando se supone dy o $\frac{dy}{dx} = 0$ se supone también cero ó nulo el ángulo que forma la tangente con el ege de las abscisas, y la tangente que es entonces *MR* ó *mD*, queda paralela á dicho ege *AB*; y como en virtud des estas suposiciones resultan los puntos *A*, *B*, *M*, *m*, que son los límites de las abscisas y ordenadas, esto es, los puntos de donde no

pueden pasar; se infiere que por un mismo método se determinan las mayores y menores abscisas y ordenadas, y los puntos que son límites de dichas abscisas y ordenadas. (p. 374)

(902. When dx or $\frac{dx}{dy}=0$ is assumed, the angle formed by the curve or its tangent with the ordinate is assumed to be zero, so that the tangent is then parallel to the ordinates as is the *AR* or *Br*; and when it is assumed dy or $\frac{dy}{dx}=0$ the angle formed by the tangent with the ege of the abscissa is also assumed to be zero, and the tangent, which is then *MR* or *mD*, remains parallel to the said axis *AB*; and since by virtue of these assumptions the points *A*, *B*, *M*, *m*, which are the limits of the abscissae and ordinates, that is, the points from which they cannot pass, it follows that by the same method the greater and lesser abscissae and ordinates, and the points which are the limits of these abscissae and ordinates, are determined).

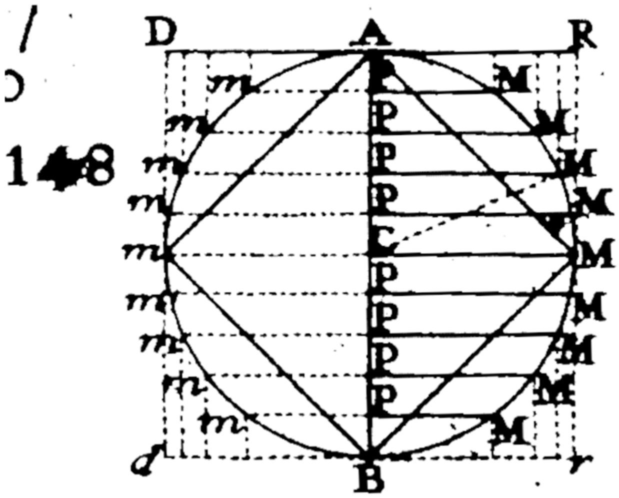


Figure 5. Graphical representation accompanying proposition 902. Source: García [57] (Plate I).

Thus, the procedure that Padilla y Arcos [55] follows to calculate the maxima and minima of the circle $y^2 = ax - x^2$ involves differentiating the equation of the circle and applying the necessary condition, which from the analytical approach, implies equating dy to zero, that is, $2y\,dy = 0 = adx - 2xdx$, whereby $x = 1/2\,a$ and substituting $y = 1/2$ a in the original equation. As we mentioned in the previous section, since the author does not consider the negative values obtained by substituting $x = 1/2\,a$ in the equation of the circumference, he only obtains one possible optimal value.

However, Bails [56] solves a similar exercise, but considering "la curva cuya equacion es $yy + xx = 2ay + 2bx - aa - bb + rr$" (p. 292) (The curve whose equation is $yy + xx = 2ay + 2bx - aa - bb + rr$). From the geometrical approach, Bails differentiates the equation obtaining $2ydy + 2xdx = 2ady + 2bdx$, simplifies the expression $dx/dy = (2y - 2a)/(2b - 2x)$ and imposes on the one side $dx = 0$ and on the other $dy = 0$. After solving both equations he finds that $y = a$ and, therefore, substituting in the equation, he obtains the two possible optimums $x = b \pm r$, and, on the other hand, $x = b$ and obtains the limits of the curve $y = a \pm r$.

4.4. Application of the Optimal Sufficient Condition

The analysis of the exercises and problems shows us that in the first examples that the authors include, the sufficient condition of maximum or minimum is always checked during the resolution. Thus, when Padilla y Arcos [54] obtains the values $x = 1/2a$ and $y = 1/2a$ when optimising the circle, he proceeds to check that $d^2y < 0$, deduces that there can only be maxima in the circle and gives, as the final solution, the maximum value

$\left(\frac{a}{2}, \frac{a}{2}\right)$. This is because he considers the equation of the circle as an implicit function, without taking into account the two branches that the equation of the circle gives us. Further, as the author does not consider negative solutions, it is not inconsistent for him to state that there is only a maximum value on the circumference at the point $\left(\frac{a}{2}, \frac{a}{2}\right)$.

Similarly, in the example in which Bails [56] calculated the maxima and minima (and limits) of the circumference, in which the equation of the circumference is considered, to verify that R and R' are limits of the curve and that T and T' are the minimum and maximum values, respectively, he substitutes values less and greater than $b - r$, $b + r$, $a - r$ and $a + r$ to verify that there are no more values of the function to the right, to the left, above or below.

In the other examples, however, none of them check that the solution obtained is really a maximum or a minimum; they all assume that the value obtained from the equation imposing the necessary condition is the final and unique solution to the problem, delegating the responsibility for the problem being well designed to the statement.

5. Discussion and Conclusions

The main objective of this work was to identify the different approaches used by the authors of textbooks on infinitesimal calculus published in Spain during the 18th century and to carry out a comparative analysis of the exercises and problems proposed, trying to identify aspects in which they were similar and different. In particular, we focused on the wording of the statements and whether they contained the necessary data to solve the problem; on the necessary conditions imposed for a point to be a maximum or a minimum; on the consideration of negative solutions or the lack of valid solutions; or on whether or not the authors applied the sufficient condition to check whether the value found was a maximum or a minimum.

The paper addressed by Blanco [47], on the *Curso militar de Mathemáticas* by Padilla y Arcos, points out that, although the author states the rules of the algorithm for solving the maxima and minima of a function and applies them in the first examples, in the successive exercises and problems proposed afterwards, he only determines whether the values obtained are maxima or minima because of the context in which the problem was posed. In this work, we have also been able to verify that the rest of the authors also use the same technique; namely, in the first examples, they apply the algorithm in detail, but in the others, they assume that the solution obtained is valid.

The analysis of the textbooks tells us that, although infinitesimal calculus was incorporated into the curricula at the end of the 18th century, the knowledge acquired about this branch of mathematics was not unified at this time and depended on the textbook or textbooks that had served as a reference for the author. Thus, the approach given to the algorithm for calculating maxima and minima influences whether the authors calculate the limits of the abscissae and the ordinates, as well as the optimal points of the function itself.

This brings us back to the importance for the teaching of mathematics in Spain of the acquisition of European reference textbooks or the training stays in prestigious European institutions, which helped the Spanish authors to organise the information and decide on the contents to be included. Needless to say, the approach that each author reflected in their texts is related to that given in the books they consulted and thanks to which they learned, taught and disseminated in Spain the advances achieved in Europe.

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References

1. Karp, A.; Furinghetti, F. *History of Mathematics Teaching and Learning: Achievements, Problems, Prospects*; Springer: Cham, Switzerland, 2016.
2. Maz, A.; Torralbo, M.; Rico, L. *José Mariano Vallejo, el Matemático Ilustrado. Una Mirada Desde la Educación Matemática*; Servicio de Publicaciones de la Universidad de Córdoba: Córdoba, Spain, 2006.
3. Karp, A. The History of Mathematics Education: Developing a Research Methodology. In *Handbook on the History of Mathematics Education*; Karp, A., Schubring, G., Eds.; Springer: Dordrecht, The Netherlands, 2014; pp. 9–24.
4. Prytz, J. Social Structures in Mathematics Education: Researching the History of Mathematics Education with Theories and Methods from the Sociology of Education. *Int. J. Hist. Math. Educ.* **2013**, *8*, 51–72.
5. Schubring, G. On the methodology of analysing historical textbooks: Lacroix as textbook author. *Learn. Math.* **1987**, *7*, 41–51.
6. Beyer, W.O. Algunos libros de Aritmética usados en Venezuela en el período 1826–1912. *Rev. Pedagog.* **2006**, *27*, 71–110.
7. Schubring, G. Computation Devices in Nineteenth-Century Mathematics Instruction in Europe. In *Computations and Computing Devices in Mathematics Education before the Advent of Electronic Calculators*; Volkov, A., Freiman, V., Eds.; Springer: New York, NY, USA, 2018; pp. 365–384.
8. Cerroni, C.; Brigaglia, A. The "Circolo Matematico di Palermo" and the First World War: The crisis of scientific internationalism: A view through the unedited correspondence of De Franchis with Edmund Landau and other mathematicians. *Hist. Math.* **2021**, *55*, 64–94.
9. Gordon Győri, J.; Fried, K.; Köves, G.; Oláh, V.; Pálfalvi, J. The Traditions and Contemporary Characteristics of Mathematics Education in Hungary in the Post-Socialist Era. In *Eastern European Mathematics Education in the Decades of Change*; Karp, A., Ed.; Springer International Publishing: Cham, Switzerland, 2020; pp. 75–129.
10. Furinghetti, F.; Giacardi, L. Secondary school mathematics teachers and their training in pre-and post-unity Italy (1810–1920). *ZDM Math. Educ.* **2012**, *44*, 537–550.
11. Amadeo, M.; Schubring, G. The École Polytechnique of Paris: Myths, sources and facts. *Bolema Bol. Educ. Mat.* **2015**, *29*, 435–451.
12. Diniz, M.A.; Bellhouse, D.R. David Gregory, John Arbuthnot and their roles in the early development of probability in Great Britain. *Br. J. Hist. Math.* **2021**, 36, 13–22.
13. Li, H.; Li, N. Features and Characteristics of Chinese New Century Mathematics Textbooks. In *The 21st Century Mathematics Education in China*; Cao, Y., Leung, F.K.S., Eds.; Springer: Berlin/Heidelberg, Germany, 2018; pp. 171–192.
14. Viertel, K. The development of the concept of uniform convergence in Karl Weierstrass's lectures and publications between 1861 and 1886. *Arch. Hist. Exact Sci.* **2021**, *75*, 455–490.
15. Ausejo, E. Quarrels of a Marriage of Convenience: On the History of Mathematics Education for Engineers in Spain. *Int. J. Hist. Math. Educ.* **2007**, *2*, 1–13.
16. Vea, F.; Velamazán, M.A. La formación matemática en la ingeniería. In *Técnica e Ingeniería en España. Volumen VI. El Ochocientos. De los Lenguajes al Patrimonio*; Silva, M., Ed.; Real Academia de Ingeniería: Zaragoza, Spain, 2011; pp. 299–344.
17. León-Mantero, C.; Maz-Machado, A.; Madrid, M.J. Juan Cortázar (1809–1873): Profesor, autor y matemático. *La Gac. La RSME* **2019**, *22*, 159–169.
18. Meavilla, V.; Oller, A. Los textos matemáticos de Antonio Terry y Rivas. *Números* **2015**, *90*, 89–103.
19. Gómez Alfonso, B.; Sanz García, M.T.; Huerta Gabarda, I. Problemas Descriptivos de fracciones. *Bolema Bol. Educ. Mat.* **2016**, *30*, 586–604.
20. Madrid, M.J.; Maz-Machado, A.; León-Mantero, C.; López-Esteban, C. Aplicaciones de las Matemáticas a la Vida Diaria en los Libros de Aritmética Españoles del Siglo XVI. *Bol. De Educ. Mat.* **2017**, *31*, 1082–1100.
21. Maz-Machado, A.; Rico, L. Principios didácticos en textos españoles de matemáticas en los siglos XVIII y XIX. *RELIME Rev. Latinoam. Investig. Educ.* **2015**, *18*, 49–76.
22. Muñoz Escolano, J.M.; Oller Marcén, A.M. Análisis de los prólogos de los textos algebraicos publicados en España durante el siglo XVI. *Hist. Y Mem. La Educ. HMe* **2020**, *11*, 51–85.
23. Puig, L. Ideas of Structure in the History of Algebra and Its Teaching. In *Algebra Structure Sense Development amongst Diverse Learners*; Rojano, T., Ed.; Routledge: New York, NY, USA, 2022; pp. 20–34.
24. Santágueda-Villanueva, M.; Gómez, B. Los modelos de enseñanza de los problemas de aligación en las enciclopedias escolares españolas. *Bolema Bol. Educ. Mat.* **2021**, *35*, 365–388.
25. Schubring, G.; Karp, A. History of Mathematics Teaching and Learning. In *Encyclopedia of Mathematics Education*, 2nd ed.; Lerman, S., Ed.; Springer Nature Switzerland AG: Cham, Switzerland, 2020; pp. 342–349.
26. Gómez, B. El análisis de manuales y la identificación de problemas de investigación en Didáctica de las Matemáticas. *PNA* **2011**, *5*, 49–65.
27. Escolano, A. El manual escolar y la cultura profesional de los docentes. *Tend. Pedagog.* **2009**, *14*, 169–180.
28. Gómez, B. La justificación de la regla de los signos en los libros de texto: ¿Por qué menos por menos es más? In *Iniciación a la Investigación en Didáctica de la Matemática. Homenaje al Profesor Mauricio Castro*; Gómez, P., Rico, L., Eds.; Universidad de Granada: Granada, Spain, 2001; pp. 257–275.

29. Ageron, P.; Hedfi, H. Ibrāhīm al-Balīshṭār's book of arithmetic (ca. 1575): Hybridizing Spanish mathematical treatises with the Arabic scientific tradition. *Hist. Math.* **2020**, *52*, 26–50.
30. Christianidis, J.; Megremi, A. Tracing the early history of algebra: Testimonies on Diophantus in the Greek-speaking world (4th–7th century CE). *Hist. Math.* **2019**, *47*, 16–38.
31. Friedman, M. On Mascheroni's La geometria del compasso at the beginning of the 19th century. *Hist. Math.* **2021**, *57*, 55–79.
32. Puig, L.; Fernández, A. La Arithmetica Algebratica de Marco Aurel, primer álgebra impresa escrita en español. Preliminares para su estudio. In *Investigación en Didáctica de la Matemática. Homenaje a Encarnación Castro*; Rico, L., Cañadas, M.C., Gutiérrez, J., Molina, M., Segovia, I., Eds.; Editorial Comares: Granada, Spain, 2016; pp. 143–150.
33. Gómez, B. La regla de dos y los problemas de acciones simultáneas. *Mat. Educ. Soc.* **2021**, *4*, 73–82.
34. Madrid, M.J.; López-Esteban, C.; Jiménez-Fanjul, N. La enseñanza de las matemáticas en la Academia de Guardiamarinas de Cádiz: Una visión a partir de tres libros clave. In *Las Matemáticas en España Durante el Siglo XVIII a Través de los Libros y Sus Autores*; Maz-Machado, A., López-Esteban, C., Eds.; Ediciones Universidad de Salamanca: Salamanca, Spain, 2020; pp. 93–113.
35. Sánchez Sierra, I.M.; González Astudillo, M.T. La geometría analítica en España durante el siglo XIX: Estudio de las soluciones negativas de una ecuación. Enseñanza de las Ciencias. *Rev. Investig. Exp. Didact.* **2017**, *35*, 89–106.
36. León-Mantero, C.; Maz-Machado, A.; Madrid, M.J. El Tratado de Álgebra elemental de Juan Cortázar: Un libro significativo para la enseñanza de las matemáticas en España. *Educ. Siglo XXI* **2021**, *39*, 235–256.
37. Ausejo, E. British influences in the introduction of calculus in Spain (1717–1767). In *The Circulation of Science and Technology: Proceedings of the 4th International Conference of the European Society for the History of Science*; Roca, A.M., Ed.; SCHCT: Barcelona, Spain, 2012; pp. 555–559.
38. Ausejo, E.; Medrano Sánchez, F.F. Jorge Juan y la consolidación del cálculo infinitesimal en España (1750–1814). In *Jorge Juan Santacilia en la España de la Ilustración*; Alberola, A., Die, R., Mas, C., Eds.; Publicacions de la Universitat d'Alacant: Alacant, Spain, 2015; pp. 155–178.
39. Hormigón, M. *Las Matemáticas en el Siglo XVIII*; Akal, S.A: Madrid, Spain, 1994; Volume 24.
40. Ausejo, E.; Medrano, F.J. Construyendo la modernidad: Nuevos datos y enfoques sobre la introducción del Cálculo Infinitesimal en España (1717–1787). *Llull Rev. Soc. Esp. Hist. Cienc. Tec.* **2010**, *33*, 25–56.
41. Cobos, J.M.; Fernández-Daza, C. *El Cálculo Infinitesimal en los Ilustrados Españoles: Francisco de Villalpando y Juan Justo García*; Universidad de Extremadura Servicio de Publicaciones: Cáceres, Spain, 1997.
42. Cuesta Dutari, N. *Historia de la Invención del Análisis Infinitesimal y de su Introducción en España*; Universidad de Salamanca: Salamanca, Spain, 1994.
43. Dorce, C. *Historia de las Matemáticas en España*; Editorial Arpegio: San Cugat, Spain, 2017.
44. Garma, S. Cultura matemática en la España de los siglos XVIII y XIX. In *Ciencia y Sociedad en España*; Sánchez, J.M., Ed.; Ediciones El Arquero: Madrid, Spain, 1988; pp. 93–127.
45. Navarro, J. Las Matemáticas en la Escuela Militar de Ávila (1774). *Gac. Real Soc. Mat. Esp.* **2011**, *14*, 309–332.
46. Navarro, J. La incorporación del cálculo diferencial e integral al Colegio de Artillería de Segovia. *Llull Rev. Soc. Esp. Hist. Cienc. Tec.* **2013**, *36*, 333–358.
47. Blanco, M. The Mathematical Courses of Pedro Padilla and Étienne Bézout: Teaching Calculus in Eighteenth-Century Spain and France. *Sci. Educ.* **2013**, *22*, 769–788.
48. García, D. Jorge Juan: El introductor del Cálculo Infinitesimal en España. *Suma Rev. Sobre Ensen. Aprendiz. Mat.* **2015**, *79*, 53–63.
49. León-Mantero, C.; Santiago, A.; Gutiérrez-Arenas, M.P. Method for finding the maxima and minima in the Eighteen Century Spanish textbooks: European influences. In *Las Matemáticas en España Durante el Siglo XVIII a Través de los Libros y Sus Autores*; López-Esteban, C., Maz-Machado, A., Eds., Universidad de Salamanca: Salamanca, Spain, 2020; pp. 115–134.
50. Furinghetti, F. History and epistemology in mathematics education. In *History of Mathematics, in Encyclopedia of Life Support Systems (e-Book)*; Hansen, V.L., Gray, J., Eds.; EOLSS: Oxford, UK, 2012.
51. Schubring, G. Researching into the History of Teaching and Learning Mathematics: The State of the Art. *Paedagog. Hist.* **2006**, *42*, 665–677.
52. Císcar, G. *Exámen Marítimo Teórico Práctico ó Tratado de Mecánica Aplicado a la Construccion*, 2nd ed.; Imprenta Real: Madrid, Spain, 1793.
53. Maz, A. Investigación histórica de conceptos en los libros de matemáticas. In *Investigación en Educación Matemática XIII*; González, M.J., González, M.-T., Murillo, J., Eds.; SEIEM: Santander, Spain, 2009; pp. 5–20.
54. Padilla y Arcos, P. *Curso Militar de Mathematicas Sobre las Partes de Estas Ciencias*; Imprenta de Antoni Marín: Madrid, Spain, 1756; Volume VI.
55. Villalpando, F. *Tractatus Praeliminaris. Matehematicorum Disciplinarum Elementa in Usum Physicae Candidatorum*; Joaquín Ibarra: Madrid, Spain, 1778.
56. Bails, B. *Elementos de Matemáticas*; Joaquín Ibarra, Madrid, Spain, 1779; Volume III.
57. García, J.J. *Elementos de Aritmética, Álgebra y Geometría*; D. Joachin Ibarra, Madrid, Spain, 1782.
58. Giannini, P. *Curso Matemático Para la Enseñanza de los Caballeros Càdetes del Real Colegio Militar de Artillería*; Oficina de Don Antonio Espinosa: Segovia, Spain, 1795; Volume III.
59. Verdejo, F. *Compendio de Matemáticas Puras y Mixtas Para Instrucción de la Juventud*; Imprenta de la Viuda de Ibarra: Madrid, Spain, 1802; Volume II.
60. McLaurin, C. *A Treatise of Fluxions in Two Books*; T.W. and T. Ruddimans: Edimburgo, UK, 1742.

61. L'Hôpital, G. *Analyse des Infiniment Petits, Pour Líntelligence des Lignes Courbes*; L'imprimerie Royale: Paris, France, 1696.
62. Euler, L. *Institutiones Calculi Differentialis cum Eius vsu in Analysi Finitorum ac Doctrina Serierum*; Academia Imperial de Ciencias: London, UK, 1755.
63. Bézout, É. *Cours de Mathématiques a L´usage des Gardes du Pavillon et la Marine*; Chez Richard, Caille & Ravier: Paris, France, 1767; Volume IV.
64. Cousin, M. *Leçons de Calcul Différentiel et de Calcul Intégral*; Claude-Antoine Jombert: Paris, France, 1777.