On the concept of infinitesimal position vector fields in Galilean spacetimes

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Abstract

We introduce two different ways to establish the concept of infinitesimal position vector field between "infinitesimally nearby" observers in a Galilean spacetime as well as show their mathematical equivalence. We also use this concept to characterize the family of spatially conformally Leibnizian spacetimes.

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1 Introduction

The first geometric model for Newton's theory of gravity, also called Newton-Cartan theory, was obtained by Cartan [4, 5] and Friedrichs [14]. This theory was developed throughout the twentieth century (see [8, 10, 11, 12, 16, 17, 19, 27]), when we can also find generalizations where Newtonian spacetimes satisfying the cosmological principle are introduced [21], obtaining the Newtonian analogs to the relativistic Robertson-Walker models.

In the current century, we can highlight the work of Bernal and Sánchez [3], where the concept of Galilean spacetime is defined and the foundations of a generalization of Newton-Cartan theory is settled in the language of modern differential geometry. Recently, de la Fuente and Rubio [7], following the line of Bernal and Sáchez, introduce the models called Galilean generalized Robertson-Walker spacetimes and study their geometrical structure as well as the completeness of its inextensible free falling observers. Also, de la Fuente and Rubio, along with Pelegrín, have recently studied the geodesic completeness of stationary Galilean spacetimes as well as the geometric conditions in these spacetimes that guarantee the existence of a global splitting as a standard stationary Galilean spacetime [6].

This geometric approach to the Newtonian gravitational theory and its generalizations continues to attract the interest of physicists and mathematicians despite the fact that at present, the general theory of relativity constitutes the best framework for the description of the universe and the gravitational phenomenon. This interest is aroused, among other reasons, by the fact that many issues considered typical of the theory of relativity are shared with the Newtonian theory after a geometric approach. Indeed, Newton-Cartan theory can be formulated as a covariant theory where gravity emerges as a manifestation of the spacetime's curvature and the spacetime's structure is dynamical in the sense that it participates in the unfolding of physics rather than being a fixed backdrop. Moreover, it allows to clarify the gauge status of the Newtonian gravitational potential [20, Sect. 4.2]. On the other hand, the geometric Newtonian aproach allows to state in coordinate-free geometric language the well-known claim that Newtonian gravitation theory (or, at least, a certain generalized version of it) is the limit of general relativity [12, 13, 18, 19]. It also has been developed to define post-Newtonian approximations to general relativity [9, 26]. Indeed, these non-relativistic models have numerous applications in condensed matter systems [25], cosmology [21], holography [1], quantum collapse [22], fractional quantum Hall effect [15] and other related phenomena.

Within this framework, we will focus on the notion of observer, which has traditionally been inherent to physics. However, if we work on a covariant (geometric) theory, one might get the false impression that all observers are physically equivalent. An immediate consequence of this error would be to deprive the concept of its meaning, since thinking that all observers are physically equivalent would make them irrelevant in the description of a physical scenario. Nevertheless, the reality is that the covariant character of a geometric theory allows its fundamental equations to be shared by all observers.

Analogously to the relativistic case, in the generalized Newton-Cartan theory, each observer can (locally) define his own coordinates via the exponential map. However, they cannot compare frame-dependent information with another observer unless they meet at the same point or come close enough so that the absolute space can be considered, effectively, as flat and then a classical Newtonnian-like situation is recovered. Since a vector space is needed to define a position vector, and therefore, the coordinates associated to it, the difference between observers in the classical theory and in generalized Newton-Cartan theory is mathematical in nature, due to the fact that a differentiable manifold does not have vector space structure in general. In order to talk about coordinates in a general n-dimensional differentiable manifold we need to focus on a localized region of the manifold, a chart, diffeomorphic to $\mathbb{R}^n$. Thus, in general, observers cannot set up reference frames to explore the whole spacetime, making the observer's role strongly local.

Nevertheless, global inferences are not impossible despite the previous comments on observers using the key concept of symmetry. Indeed, if the quantities that we are measuring follow a pattern, then the whole spacetime does not need to be explored since a local study can be extrapolated to figure out global properties of the spacetime.

In this work, following the terminology given in [3, 6, 7], we devote Section 3 to establish in a accurate mathematical form the notions of infinitesimal relative position vector field and neighbor vector field with respect to a fixed observer in a congruence or vector field of observers (Definitions 2 and 3, respectively). Moreover, we show the mathematical equivalence of these two notions in Theorem 4. Additionally, in Section 3.1 we provide some physical interpretations of an observer's infinitesimal position vectors. Finally, we make use of these concepts in Section 4 to characterize the family of conformally Leibnizian spacetimes introduced in [7] in terms of a field of observer's infinitesimal position vector fields in Theorems 9 and 14.

2 Set up

A *Leibnizian* spacetime is a triad (M, Ω, g), consisting of a smooth connected manifold M of arbitrary dimension $m = n + 1 \geq 2$, a nowhere vanishing differential 1-form $\Omega \in \Lambda^1(M)$ ($\Omega_p \neq 0$, $\forall p \in M$), and a smooth, bilinear, symmetric and positive definite map

$$g : \Gamma(\mathrm{An}(\Omega)) \times \Gamma(\mathrm{An}(\Omega)) \longrightarrow C^\infty(M),\ (V, W) \mapsto g(V, W),$$

where $\mathrm{An}(\Omega) = \{v \in TM\ ,\ \Omega(v) = 0\}$ is the smooth n-distribution induced on M by Ω and the symbol Γ denotes the corrresponding vector fields, so $\Gamma(\mathrm{An}(\Omega)) = \{V \in \Gamma(TM) \,/\, V_p \in \mathrm{An}(\Omega), \forall p \in M\}$. Hence, M is endowed with a Riemannian vector bundle $(\mathrm{An}(\Omega), g)$. The pair (Ω, g) is called a Leibnizian structure. See [2] and [3] for details.

The points of M are usually called *events*. The Euclidean vector space $(\mathrm{An}(\Omega_p)\,, g_p)$ is called the *absolute space* at $p \in M$ and the linear form Ω_p is the *absolute clock* at p. A tangent vector $v \in T_pM$ is said to be *spacelike* if $\Omega_p(v) = 0$ and, otherwise, *timelike*. Additionally, if $\Omega_p(v) > 0$ (resp. $\Omega_p(v) < 0$), v points towards the *future* (resp. the *past*).

An *observer* in a Leibnizian spacetime M is a unitary future pointing timelike smooth curve $\gamma : J \longrightarrow M$, i.e., its velocity γ' satisfies that $\Omega(\gamma'(s)) = 1$ for all $s \in J$. The parameter s is called the *proper time* of the observer γ. A vector field $Z \in \Gamma(TM)$ with $\Omega(Z) = 1$ is called a *field of observers*, this is, its integral curves are observers.

When the smooth distribution $\mathrm{An}(\Omega)$ is completely integrable (equivalently, if the absolute clock Ω satisfies $\Omega \wedge d\Omega = 0$), the Leibnizian spacetime (M, Ω, g) is said to be *locally synchronizable*, and making use of Frobenius' Theorem (see [28]), it can be foliated by a family of spacelike hypersurfaces $\{\mathcal{F}_\lambda\}$. In this case, it is well-known that each $p \in M$ has a neighborhood U where $\Omega\big|_U = \beta\, dt$, for certain smooth functions β, $t \in C^\infty(U)$, $\beta > 0$, and the hypersurface $\{t = \text{constant}\}$ locally coincides with a leaf of the foliation $\mathcal{F}$. Thus, any observer may be synchronized through the "compromise time" t, obtained rescaling its proper time. In the more restrictive case $d\Omega = 0$, the Leibnizian spacetime (M, Ω, g) is called *proper time locally synchronizable*, and, locally, $\Omega = dt$. Now, observers are synchronized directly by its proper time (up to a constant). When Ω is exact, $\Omega = dT$ for some function $T \in C^\infty(M)$, called the *absolute time function*. In this case, any observer may be assumed to be parametrized by T. Notice that the notion of (local and global) synchronizability is intrinsic to the Leibnizian structure, applicable for any observer, in contrast to the relativistic setting, where the analogous concepts only have meanings with respect to fields of observers.

According to [3], a vector field is called *Leibnizian* or *rigid* if the stages Φ_s of its local flows are *Leibnizian diffeomorphisms*, that is, they preserve the absolute clock and space, i.e.,

$$\Phi_s^*\Omega = \Omega \qquad \text{and} \qquad \Phi_s^* g = g.$$

Equivalently, $L_K\Omega = 0$ and $L_K g = 0$. Leibnizian vector fields are also characterized by

$$\Omega([K,X]) = K(\Omega(X)) \quad \forall X \in \Gamma(TM) \text{ and}$$
$$K(g(V,W)) = g([K,V],W) + g(V,[K,W]) \quad \forall V,W \in \Gamma(\mathrm{An}(\Omega)).$$

On the other hand, the inertia principle must be codified through a connection on the spacetime. However, a Leibnizian structure does not have a canonical affine connection. Therefore, it is required to introduce a compatible connection with the absolute clock Ω and the space metric g, i.e., a connection ∇ such that

(a) $\nabla\Omega = 0$ (equivalently, $\Omega(\nabla_X Y) = X(\Omega(Y))$ for any $X, Y \in \Gamma(TM)$).

(b) $\nabla g = 0$ (i.e., $Z(g(V,W)) = g(\nabla_Z V, W) + g(\nabla_Z W, V)$ for any $Z \in \Gamma(TM)$ and V, W spacelike vector fields).

Such a connection is named *Galilean* and its restriction to each spacelike leaf of the foliation $\mathcal{F}$ coincides with the Levi–Civita connection associated to g. A *Galilean spacetime* (M, Ω, g, ∇) is a Leibnizian spacetime endowed with a Galilean connection ∇. In addition, ∇ is *symmetric* if its torsion vanishes identically ($\mathrm{Tor}_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \equiv 0$). From a physical point of view, a symmetric connection is desirable since it is completely determined by its geodesics, in fact, by the free falling observers of M.

Given a Galilean spacetime, its connection ∇ induces a connection along any observer γ. Its corresponding covariant derivative is given by

$$\frac{DY}{dt} = \nabla_{\gamma'(t)} Y,$$

for any vector field Y along γ. The covariant derivative of $\gamma\,'$, $\frac{D\gamma\,'}{dt}$, is understood as the (proper) *acceleration* of the observer γ. Notice that $\frac{DY}{dt} = \nabla_{\gamma'(t)} Y \in \Gamma(\mathrm{An}(\Omega))$ for any $Y \in \Gamma(\mathrm{An}(\Omega))$. Therefore the parallel transport along γ,

$$P^\gamma_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M,$$

satisfies

$$P^{\gamma}_{t_1,t_2}(\mathrm{An}(\Omega_{\gamma(t_1)})) = \mathrm{An}(\Omega_{\gamma(t_2)}).$$

We can also consider the covariant derivative along any curve $\alpha(s)$ in M. This curve will be called a geodesic if $\dfrac{D\alpha'}{ds} \equiv 0$. Notice that if α is a geodesic, then $\Omega(\alpha')$ is constant along α, and so the causal character is conserved.

For each fixed field of observers Z on a Galilean spacetime (M, Ω, g, ∇), the *gravitational field* induced by ∇ in Z is given by the spacelike vector field $\mathcal{G} = \nabla_Z Z$. The *vorticity* or *Coriolis field* of Z is the 2−form $\omega(Z) = \frac{1}{2}\mathrm{Curl}(Z)$, defined as

$$\omega(Z)(V, W) = \frac{1}{2}\Big(g(\nabla_V Z, W) - g(\nabla_W Z, V)\Big) \qquad \forall V, W \in \Gamma(\mathrm{An}(\Omega)). \tag{1}$$

Z is called *irrotational* when ω vanishes. This definition can be extended to vector fields as long as $\Omega(\nabla_V Z) = 0$ for all $V \in \Gamma(\mathrm{An}(\Omega))$.

It has been proven that, for a fixed field of observers Z on a Leibnizian spacetime (M, Ω, g), the set of all Galilean connections is bijectively mapped onto

$$\Gamma(TM) \times \Lambda^2(\mathrm{An}(\Omega)) \times \Lambda^2\big(TM, \mathrm{An}(\Omega)\big),$$

see [3, Th.27]. Each Galilean connection ∇ is mapped to $\Big(\mathcal{G}(Z), \omega(Z), P^Z \circ \mathrm{Tor}\Big)$, where

$$P^Z : \Gamma(TM) \longrightarrow \Gamma\big(\mathrm{An}(\Omega)\big), \qquad P^Z(X) = X - \Omega(X)\, Z, \quad \forall X \in \Gamma(TM). \tag{2}$$

In particular, $d\Omega = \Omega \circ Tor$, [3, Lemma 13]. Thus, the existence of a symmetric Galilean connection for a Leibnizian structure implies the proper time local synchronizability of the latter and each symmetric Galilean connection is uniquely determined by the gravitational field and the vorticity of the field of observers Z.

Additionally, a Leibnizian vector field K in a Galilean spacetime (M, Ω, g, ∇) is named *Galilean* if it is affine for ∇, that is, $L_K \nabla = 0$, where L denotes the Lie derivative. Equivalently, if

$$[K, \nabla_Y X] = \nabla_{[K,Y]} X + \nabla_Y [K, X], \quad \forall X, Y \in \Gamma(TM).$$

Finally, a Galilean spacetime with symmetric connection ∇ is called *Newtonian* if ∇ restricted to the spacelike vectors is flat and if it admits an irrotational Galilean field of observers. This kind of spacetimes has traditionally represented the classical (non relativistic) geometric models of gravity.

3 Position vector fields in Galilean spacetimes

Let (M, Ω, g, ∇) be a locally synchronizable Galilean spacetime, and Z a field of observers in M. From the integrability of $\mathrm{An}(\Omega)$, at some neighborhood of each p, U, there exists a chart $(U; t, x_1, \ldots, x_n)$, $t(p) = 0$, and a smooth function $\beta : U \longrightarrow \mathbb{R}$, $\beta > 0$, such that

$$\Omega\big|_U = \beta\, dt\,, \qquad Z\big|_U = R + \frac{1}{\beta}\partial_t\,, \qquad \text{and} \qquad \Omega(\partial_{x_i}) = 0 \;\; \forall i = 1, \ldots, n,$$

where R is a spacelike vector field on U and $\partial_t \equiv \frac{\partial}{\partial t}$.

The coordinate function t is a (local) *compromise time* for the observers of Z, and each spacelike hypersurface (i.e., hypersurface in which every tangent vector is spacelike) $\mathcal{F}_{t_0} \equiv \{t = t_0\}$ is considered as their "rest space" when the compromise time is t_0.

Let us consider one of the observers of Z, $\gamma : I \subseteq \mathbb{R} \longrightarrow M$, and we reparametrize it such that $t(\gamma(0)) = 0$. The compromise time along γ is related with its proper time through the expression

$$t \circ \gamma\,(s) = \int_0^s \frac{dl}{\beta \circ \gamma\,(l)}. \tag{3}$$

Remark 1 In the particular case of $d\Omega = 0$, the coordinate function t could be chosen such that $\Omega\big|_U = dt$ and $Z = \partial_t$. Moreover, since $\Omega(\gamma'(s)) = 1$ for any $s \in I$, we have that $s = t \circ \gamma(s)$, i.e., the observer γ is parametrized by the (local) absolute time t.

Let us recall the exponential map at a point $p \in M$ given by $\exp_p(v) = \alpha_v(1)$ for all $v \in T_pM$ such that the unique inextensible geodesic α_v in M with initial velocity v is defined on the interval $[0, 1]$. From the properties of the exponential map, it may be found $J \subseteq I$, $0 \in J$, and a family of open subsets $E_s \subseteq \mathrm{An}\big(\Omega_{\gamma(s)}\big)$ such that the map

$$\bigsqcup_{s \in J} E_s \longrightarrow \mathcal{S} = \bigcup_{s \in J} \mathcal{S}_s \subseteq M, \qquad (s, v) \longmapsto \exp_{\gamma(s)}(v),$$

is a diffeomorphism [23, Prop. 5.18], being $\mathcal{S}_s = \exp_{\gamma(s)}(E_s) \cap U$. Moreover, since for any geodesic $\alpha(s)$ we have $\frac{D}{ds}(\Omega(\alpha')) = \Omega(\nabla_{\alpha'}\alpha') = 0$, the causal character is conserved along geodesics. In particular, geodesics with spacelike initial vector are spacelike (i.e., its tangent vector is spacelike at any point). Therefore, $\mathcal{S}_s \subseteq \mathcal{F}_{t(\gamma(s))}$.

Now, consider another observer σ in Z, next to γ, with $\sigma(0) \in \mathcal{S}_0$. Thus, for each $q \in \mathrm{Im}(\sigma) \cap \mathcal{S}$, there exists a unique spacelike vector in $\bigsqcup_{s\in J} E_s$, $Q(s)$, such that $q = \exp_{\gamma(s)}(Q(s))$. This vector field $Q(s)$, will be called the *(finite) position vector field of σ with respect to γ at the instant $s \in J$*, and the quantity

$$d_{\gamma(s),\sigma} = \sqrt{g(Q(s), Q(s))},$$

will be said to be the *distance of the observer σ measured by γ at the instant $s \in J$*.

Notice that this procedure describes a new parametrization of the worldlines of the observers of Z close to γ and, as consequence, a new local timelike vector field $\overline{Z}$ satisfying $\overline{Z}_{\gamma(s)} = Z_{\gamma(s)}$. We denote by Ψ_s the local flux of $\overline{Z}$, which does preserve the spacelike character along γ, i.e., if v is a spacelike vector in $p = \gamma(0)$, then $d\Psi_s\big|_p(v)$ is spacelike for any $s \in J$.

Let us introduce an infinitesimal notion of the position vector of the neighboring observers of Z with respect to γ.

Definition 2 *Let $v \in \mathrm{An}(\Omega_{\gamma(0)})$ be a spacelike vector in the wordline of γ. We define the* infinitesimal position vector field associated to v with respect to γ *as the only Ψ-invariant vector field along γ with $V(0) = v$,*

$$V(s) = d\Psi_s\big|_{\gamma(0)}(v).$$

Note that V is Lie-parallel with respect to $\overline{Z}$, i.e, $L_{\overline{Z}}V(s) = 0$, for any $s \in J$.

We can characterize now the infinitesimal position vector fields. First, notice the existence of a positive function $h \in C^\infty(M)$ with $(h \circ \gamma)(s) = 1$, for all $s \in J$, such that

$$\overline{Z} := h\, Z.$$

Given an infinitesimal position vector field $V \in \mathrm{An}\big(\Omega|_\gamma\big)$, a vector field $\widetilde{V}$ may be defined on a neighborhood of γ such that $\widetilde{V}_{\gamma(s)} = V(s)$ and $[\overline{Z}, \widetilde{V}] = 0$. Hence, from the spatial character of $\widetilde{V}$ along γ, a direct computation shows that

$$0 = \overline{Z}\big(\Omega(\widetilde{V})\big)\big|_\gamma = \Big(\Omega\big(\nabla_{\widetilde{V}}\overline{Z}\big) + \Omega \circ \mathrm{Tor}(\overline{Z}, \widetilde{V})\Big)\,\big|_\gamma =$$
$$\Big(\widetilde{V}(h) + h\, d\Omega(Z, \widetilde{V})\Big)\,\big|_\gamma = V(h) + d\Omega(\gamma', V). \tag{4}$$

There is another natural way to describe the relative position of nearby observers to a given one using the concept of neighbor vector field (see [24, Def. 2.3.2]).

Definition 3 *Given a field of observers Z, a vector field W along an observer γ in Z is called a* neighbor vector field of γ in Z *if there exists a vector field $\widetilde{W}$ along γ such that*

(i) $P^Z\widetilde{W} = W$, *where P^Z is defined in (2),*

(ii) $L_Z\widetilde{W} = 0$ *(i.e., $[Z, \widetilde{W}] = 0$).*

It is clear that, given a spacelike vector v in $p = \gamma(0)$, there exists a unique neighbor vector field of γ in Z such that $W(0) = v$. The following result shows the equivalence between the notion of infinitesimal relative position vector field and neighbor vector field.

Theorem 4 *Let Z be a field of observers, $\gamma : I \longrightarrow M$ an observer of Z and v a spacelike vector in $p = \gamma(0)$. A vector field W along γ with $W(0) = v$ is the only neighbor vector field along γ satisfying $W(0) = v$ if and only if it is the only infinitesimal position vector field for γ satisfying $W(0) = v$.*

Proof. Let $\widetilde{W}$ be a vector field defined on a neighborhood $U \subseteq M$ of γ. Let us denote $P^Z\widetilde{W} = \widetilde{V}$. We may decompose $\widetilde{W} = \widetilde{V} + f\,Z$ for certain $f \in C^\infty(U)$, obtaining

$$[Z, \widetilde{W}] = [Z, \widetilde{V}] + Z(f)\,Z.$$

Thus, taking into account that $\overline{Z} = h\,Z$, we have

$$[Z, \widetilde{W}] = (1/h)[\overline{Z}, \widetilde{V}] + (Z(f) + \widetilde{V}(h)/h)Z.$$

Considering a function f satisfying $Z(f) = -\widetilde{V}\big(\ln(h)\big)$, the result follows.

□

Notice that (4) provides a new characterization of the infinitesimal vector fields along γ in Z.

Proposition 5 *Let Z be a field of observers and $\gamma : I \longrightarrow M$ one of its integral curves. Then, the flux of $h\,Z$ defines (infinitesimal) position vector fields with respect to γ if and only if, for some $J \subseteq I$, h satisfies*

(a) $(h \circ \gamma)(s) = 1, \quad \forall s \in J.$

(b) $dh\big|_{\gamma(s)} = -L_Z \Omega\big|_{\gamma(s)}, \quad \forall s \in J.$

Remark 6 In a locally proper time synchronizable Leibnizian spacetime, the flux ψ_s coincides with the flux of Z. Therefore, $h \equiv 1$ and the infinitesimal position vector fields of an observer γ in Z are simply the Z-invariant spacelike vector fields along γ.

Using of the local compromise time t in the spacetime, we may obtain a local expression of the previous function h. Let $\sigma : I_\sigma \longrightarrow M$ be another wordline of Z with $\sigma(0) \in \mathcal{S}_0 \subseteq \mathcal{F}_{\gamma(0)}$. When its clock marks a value u (supposing that $\sigma([0,u]) \subseteq U$), the compromise time t is

$$t \circ \sigma(u) = \int_0^u \frac{1}{\beta \circ \sigma(l)} dl,$$

whereas the clock of γ will mark $s = (t \circ \gamma)^{-1}(t \circ \sigma(u))$. Therefore,

$$u(s) = (t \circ \sigma)^{-1} \circ (t \circ \gamma)(s).$$

Thus,

$$\overline{Z}_{\sigma(u)} = \frac{\beta \circ \sigma(u)}{\beta \circ \gamma \circ \big((t \circ \gamma)^{-1} \circ (t \circ \sigma(u))\big)} Z_{\sigma(u)}.$$

Hence, the function h is locally given by

$$h\big|_U = \frac{\beta}{\beta \circ \gamma \circ (t \circ \gamma)^{-1} \circ t}.$$

3.1 Physical interpretation of infinitesimal position vector fields

The modulus of the infinitesimal position vector field $V(s)$ may be interpreted as a measure of the distance from γ to the close neighbor observers in Z. Therefore, the covariant derivative $\frac{DV}{ds} = \nabla_{\gamma'} V$ represents the velocity with respect to γ of its close neighbors. For a field of observers Z, let us denote

$$A_Z : \Gamma\big(\mathrm{An}(\Omega)\big) \longrightarrow \Gamma\big(\mathrm{An}(\Omega)\big), \qquad A_Z(V) = -\nabla_V Z, \quad \forall V \in \Gamma\big(\mathrm{An}(\Omega)\big).$$

Observe that, since every infinitesimal position vector field along γ, $V(s)$, is invariant along the flux of Z, we have

$$\frac{DV}{ds} = \nabla_{\gamma'} V = \nabla_V \overline{Z}\big|_\gamma + \mathrm{Tor}(\overline{Z}, \widetilde{V})\big|_\gamma = -A_Z(V)\big|_\gamma - V(h)\,\gamma' + \mathrm{Tor}(\gamma', V). \quad (5)$$

Operating by the left with P^Z in (5) we obtain

$$\frac{DV}{ds} = -A_Z(V) + P^Z \circ \mathrm{Tor}(\gamma', V). \quad (6)$$

If the Galilean connection is symmetric, the last member in (6) vanishes, having

$$\frac{DV}{ds} = -A_Z(V). \quad (7)$$

Until the end of the section, we will focus on this case. The linear operator A_Z may be decomposed in its symmetric $\widehat{S}$ and skew-symmetric $\widehat{\omega}$ parts as

$$-A_Z = \widehat{S} + \widehat{\omega},$$

where $\widehat{S}$ is self-adjoint for g and $\widehat{\omega}$ skew-adjoint. Let us denote by S and ω the corresponding fields of 2-covariant associated tensors

$$S(V, W) = g(\widehat{S}(V), W) = \frac{1}{2}\big(g(\nabla_V Z, W) + g(\nabla_W Z, V)\big) = \gamma'(g(V, W)), \quad (8)$$

$$\omega(V, W) = g(\widehat{\omega}(V), W) = \mathrm{Curl}(Z)(V, W), \quad (9)$$

where V, W are spacelike vector fields in M and ω is the vorticity or Coriolis tensor field defined in (1). Note that if Z represents the wordlines of the particles of a fluid and γ is the trajectory of one of them, $\omega(\gamma')$ measures how the others turn around, which justifies calling this tensor field "vorticity". The name of "Coriolis field" arises because ω measures the lack of inerciality of Z due to the observers' spinning.

On the other hand, the symmetric operator $\widehat{S}$ can be decomposed as

$$\widehat{S} = \frac{\operatorname{div}(Z)}{n} I + \Theta,$$

where I denotes the identity endomorphism of $\Gamma\big(\mathrm{An}(\Omega)\big)$ and Θ is the *shear tensor* (the traceless part of $\widehat{S}$). The term $\frac{1}{n}\operatorname{div}(Z) = \frac{1}{n}\mathrm{Trace}(A_Z)$ represents the *expansion*, i.e., it measures how neighboring observers go away on average from a fixed observer, whereas Θ measures the deviations from this average.

Remark 7 The operator $\widehat{S}$ is intrinsic to the Leibnizian structure and does not depend on the Galilean connection, see [3, Prop. 23]. This can be proved taking into account that, for any spacelike vector V,

$$S(V,V) = -g([Z,V],V) + \frac{1}{2} Z\big(g(V,V)\big).$$

In the same proposition it is proven that Z is Leibnizian if and only if $\widehat{S} \equiv 0$, (see [3, Prop. 23]).

From the expression

$$d\Omega(Z,Y) = Z(\Omega(Y)) - Y(\Omega(Z)) - \Omega([Z,Y]), \ \forall Y \in \Gamma(TM), \tag{10}$$

we easily obtain that in the proper time locally synchronizable case $L_Z\Omega = 0$, i.e., the field of observers has a Leibnizian behavior relative to the clock. And so, $\widehat{S} \equiv 0$ if and only if the local flows of Z preserve the space.

4 Infinitesimal position vector fields in conformally Leibnizian spacetimes

In this section we will study the properties and interpretations of the operator A_Z when the field of observers Z is conformally stationary/rigid. First of all, let us recall the concept of spatially conformally Leibnizian vector field, introduced in [7], which generalizes the notion of Leibnizian observer.

Definition 8 *Let (M, Ω, g) be a Leibnizian spacetime and K a vector field such that $\Omega([K, V]) = 0$ for all $V \in \Gamma(\mathrm{An}(\Omega))$. The vector field K is a* spatially conformally Leibnizian *vector field if the Lie derivative of the absolute space metric satisfies*

$$L_{_K} g = 2\lambda\, g, \tag{11}$$

for some smooth function $\lambda \in C^\infty(M)$. If K additionally verifies

$$L_{_K} \Omega = \lambda\, \Omega, \tag{12}$$

for the same conformal factor λ, then K is named conformally Leibnizian vector field.

Our next result characterizes the behaviour of the infinitesimal position vector fields associated to an irrotational and spatially conformally Leibnizian field of observers.

Theorem 9 *Let Z be a field of observers in a Galilean spacetime (M, Ω, g, ∇) with symmetric connection.*

(a) If Z is an irrotational and spatially conformally Leibnizian field of observers, $\gamma : I \longrightarrow M$ is one of its observers and $V(s)$ is an infinitesimal position vector field along γ, then the following relation holds,

$$\frac{DV}{ds}(s) = \frac{1}{n}\mathrm{div}(Z)\, V(s), \quad \forall s \in I. \tag{13}$$

(b) Conversely, if every infinitesimal position vector field $V(s)$ along any observer γ associated to Z satisfies

$$\frac{DV}{ds}(s) = \lambda\; V(s), \quad \forall s \in I,$$

for some smooth function $\lambda \in C^\infty(M)$, then Z is an irrotational and spatially conformally Leibnizian vector field with conformal factor $\lambda = \frac{1}{n}\mathrm{div}(Z)$.

Proof.

From the symmetry of ∇, we get $d\Omega = 0$. Therefore, from (10), we can easily see that the field of observers Z satisfies

$$\Omega([Z, W]) = 0, \quad \forall W \in \Gamma\big(\mathrm{An}(\Omega)\big).$$

Consequently, assumption (11) may be expressed for our spatially conformally Leibnizian field of observers Z as

$$Z(g(V,W)) - g([Z,V],W) - g([Z,W],V) = 2\lambda\, g(V,W), \quad \forall V,W \in \Gamma(\mathrm{An}(\Omega)). \tag{14}$$

Moreover, we can write (14) as

$$g(\nabla_V Z, W) + g(\nabla_W Z, V) = 2\lambda\, g(V,W), \quad \forall V,W \in \Gamma(\mathrm{An}(\Omega)). \tag{15}$$

Taking into account the irrotational character of Z in (15), we obtain

$$g(\nabla_V Z, W) = \lambda\, g(V,W), \quad \forall V,W \in \Gamma(\mathrm{An}(\Omega)). \tag{16}$$

Thus, for an infinitesimal position vector field $V(t)$ along γ we deduce from (7) and (16)

$$\frac{DV}{ds} = -A_Z(V) = \lambda V = \frac{1}{n}\mathrm{div}(Z)V.$$

To prove (b), it suffices to check out that Z is irrotational, and therefore, in the previous computations all the necessary conditions are also sufficient as well as noticing than if condition (11) is satisfied for every infinitesimal position vector field, it holds for every spacelike vector.

□

This result physically means that γ measures that every neighboring observer of Z is approaching or moving away (depending on the sign of $\mathrm{div}(Z)|_{\gamma(t)}$) along its position direction.

Remark 10 Taking into account [7, Remark 7], it is clear that the only conformally Leibnizian fields of observers in a proper time locally synchronizable Leibnizian spacetime are the Leibnizian (rigid) field of observers.

We recall that a Galilean spacetime with symmetric connection admitting a timelike vector field $K \in \Gamma(TM)$ such that

$$\nabla_X K = \rho X, \quad \forall X \in \Gamma(TM), \tag{17}$$

is called *Irrotational Conformally Leibnizian spacetime* (see [7, Def. 9]).

As a direct consequence of (17), the vector field K is conformally Leibnizian and $\mathrm{Curl}(K)(V,W) = 0$ for all spacelike vector fields V, W.

When $\rho = 0$, the vector field K is Leibnizian and so, the spacetime is called *irrotational Leibnizian spacetime*. Note that in this case $\Omega(K)$ is a constant fuction, so we can assume that the vector field K is a field of observers.

Corollary 11 *A Galilean spacetime with symmetric connection is an irrotational Leibnizian spacetime if and only if there exists a field of observers Z such that every infinitesimal position vector field V along any observer γ associated to Z satisfies*

$$\frac{DV}{ds}(s) = 0, \quad \forall s \in I.$$

Example 12 For instance, consider the Galilean spacetime $(I \times S, dt, \pi_S^* g_S, \nabla)$, where $I \subseteq \mathbb{R}$ is an interval of the real line, (S, g_S) is an n-dimensional connected Riemannian manifold, $t : I \times S \longrightarrow I$ and $\pi_S : I \times S \longrightarrow S$ denote the canonical projections, and ∇ is the unique symmetric connection verifying $\nabla_{\partial_t} \partial_t = 0$ and $\mathrm{Curl}(\partial_t) = 0$. This model is a standard static Galilean spacetime in the sense of [6, Def. 7] and it is easy to see that ∂_t is a Leibnizian field of free falling observers such that every infinitesimal position vector field V along any observer associated to ∂_t satisfies $\frac{DV}{ds}(s) = 0$ for $s \in \mathbb{R}$.

Taking into account the local structure of an irrotational (conformally) Leibnizian spacetime [7, Th. 12] and the Koszul formula given in [3, Lemma 25], it is not difficult to see that the field of observers Z is an affine field for ∇. Consequently, in the assumptions of previous corollary, the spacetime (M, Ω, g, ∇) is locally a standard static Galilean spacetime (see [6, Def. 7]).

Corollary 13 *Let (M, Ω, g, ∇) be a Galilean spacetime with symmetric connection. If there exists a field of observers Z such that for each infinitesimal position vector field V along any observer the equality $\frac{DV}{ds} = 0$ holds, then (M, Ω, g, ∇) is locally a standard static Galilean spacetime.*

At this point, the following geometric question arises. If Z is a field of observers in a Leibnizian spacetime M, under which conditions on the geometry of M and Z does there exist a function φ such that φZ is a conformally Leibnizian vector field?

Theorem 14 *Let Z be a field of observers in a Galilean spacetime (M, Ω, g, ∇) with symmetric connection such that $\frac{DV}{ds}$ is proportional to V for any infinitesimal position vector field V along any observer. Then,*

(a) the equality

$$\Omega \wedge d\big(\mathrm{div}(Z)\big) = 0 \tag{18}$$

holds if and only if for any point $p \in M$ there exists a neighborhood U_p, and a smooth function $\varphi \in C^\infty(U_p)$, such that $\varphi\, Z$ is conformally Leibnizian.

(b) if M is deformable to a point, equation (18) is equivalent to the existence of a global function $\varphi \in C^\infty(M)$ such that $\varphi\, Z$ is conformally Leibnizian.

Proof. Denote by $\theta(Z) = \mathrm{div}(Z)$. From condition (18), the 1-form $\theta(Z)\,\Omega$ is closed, i.e., locally exact. Thus, locally, there exists a positive function φ such that $d(\ln(\varphi)) = \theta(Z)\,\Omega$. Therefore, if W is a spacelike vector field, it follows that $W(\ln(\varphi)) = 0$, i.e., φ is spatially constant, and therefore, $\Omega([\varphi Z, W])$ also vanishes. Consequently, a direct computation shows that, for any $W \in \Gamma\big(\mathrm{An}(\Omega)\big)$,

$$L_{\varphi Z}\, g(W,W) = 2\,\varphi \frac{\theta(Z)}{n}\, g(W,W),$$

where we have taken into account that ∇ is symmetric and, by hypothesis, $-A_Z(W) = \nabla_W Z = \frac{\theta(Z)}{n}\, W$. This implies that, locally, $\varphi\, Z$ is spatially conformally Leibnizian with conformal factor $\frac{\theta(Z)}{n}$. To complete the necessary condition,

$$L_{\varphi Z}\, \Omega(X) = X(\varphi) = d\varphi(X) = \varphi\theta(Z)\Omega(X),$$

for every vector field X. The converse is proved in the same way. Finally, the topological assumption on M and Poincaré's lemma allow us to obtain the global result in (b).

□

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References

[1] R. Andringa, E. Bergshoeff, S. Panda, and M. de Roo, Newtonian gravity and the Bargmann algebra, *Classical Quant. Grav.*, **28** (2011), 105011 (1–12).

[2] A.N. Bernal, M. López and M. Sánchez, Fundamental units of length and time, *Found. Phys.*, **32** (2002), 77–108.

[3] A.N. Bernal and M. Sánchez, Leibnizian, Galilean and Newtonian structures of space-time, *J. Math. Phys.*, **44** (2003), 1129–1149.

[4] E. Cartan, Les varièt́es a conexion affine, *Ann. Ec. Norm. Sup.*, **40** (1923), 1–25.

[5] E. Cartan, Les varièt́es a conexion affine (suite), *Ann. Ec. Norm. Sup.*, **41** (1924), 325–412.

[6] D. de la Fuente, J.A.S. Pelegrín and R.M. Rubio, On the geometry of stationary Galilean spacetimes, *Gen. Relat. Gravit.*, **53** (2021), 8 (1–15).

[7] D. de la Fuente and R.M. Rubio, Galilean Generalized Robertson-Walker spacetimes: a new family of Galilean geometrical models, *J. Math. Phys.*, **59** (2) (2018), 022903.

[8] G. Dautcourt, Die Newtonische Gravitationstheorie als strenger Grenzfall der allgemeinen Relativitätstheorie, *Acta Phys. Pol.*, **65** (1964), 637–646.

[9] G. Dautcourt, Post-Newtonian extension of the Newton - Cartan theory, *Classical Quant. Grav.*, **14** (1997), A109–A117.

[10] W. G. Dixon, On the uniqueness of the Newtonian theory as a geometric theory of gravitation. *Commun. Math. Phys.*, **45**, (1975) 167–182.

[11] H. Dombrowski and K. Horneffer, Die Differentialgeometrie des Galileischen Relativitätsprinzips, *Math. Z.*, **86** (1964), 291–31.

[12] J. Ehlers, Über den Newtonschen Grenzwert der Einsteinschen Gravitationstheorie, *Grundlagen Probleme der Modernen Physik*, Wissenschaftsverlag, 1981.

[13] J. Ehlers, Republication of: On the Newtonian limit of Einstein's theory of gravitation, *Gen. Relat. Gravitation*, **51** (2019), 163 (1–20).

[14] K. Friedrichs, Eine Invariante Formulierung des Newtonschen Gravitationsgesetzes und der Grenzüberganges vom Einsteinschen zum Newtonschen Gesetz, *Math. Ann.*, **98** (1927), 566–565.

[15] M. Geracie, D.T. Son, C. Wu and S.F. Wu, Spacetime symmetries of the quantum Hall effect, *Phys. Rev. D*, **91** (2015), 045030 (1–18).

[16] P. Havas, Four-dimensional formulations of Newtonian mechanics and their relation to the special and the general theory of relativity, *Rev. Mod. Phys.*, **36** (1964), 938–965.

[17] H.P. Künzle, Galilei and Lorentz structures on space-time: comparison of the corresponding geometry and physics, *Ann. I. H. Poincaré A*, **17** (1972), 337–362.

[18] H.P. Künzle, Covariant Newtonian limts of Lorentz space-times, *Gen. Relat. Gravitation*, **7** (1976), 445–457.

[19] M. Lottermoser, Über den Newtonschen Grenzwert der Allgemeinen Relativitätstheorie und die relativistische Erweiterung Newtonscher Anfangsdaten, unpublished dissertation, 1988.

[20] D.B. Malament, Topics in the Foundations of General Relativity and Newtonian Gravitation Theory, *Chicago Lectures in Physics CLP*, 2012.

[21] F. Müler-Hoissen, The cosmological principle and a generalization of Newton's theory of gravitation, *Gen. Relat. Gravitation*, **15** (1983), 1051–1066.

[22] R. Penrose, On gravity's role in quantum state reduction, *Gen. Relat. Gravit.* **8**, (1996) 581–600.

[23] P. Petersen, *Riemannian Geometry*, Springer, 2006.

[24] R.K. Sachs and H. Wu, *General Relativity for Mathematicians*, Graduate Texts in Math., **48**, Springer-Verlag, 1977.

[25] D.T. Son and M. Wingate, General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas, *Ann. Phys.*, **321** (2006), 197–224.

[26] W. Tichy and É.É. Flanagan, Covariant formulation of the post-1-Newtonian approximation to general relativity, *Phys. Rev. D*, **84** (2011), 044038 (1–14).

[27] A. Trautman, Comparison of newtonian and relativistic theories of space-time, *Perspectives in geometry and relativity*, Indiana University Press, 1966.

[28] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Grad. Texts Math., Springer-Verlag, 1983.