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LAYER-AVERAGED APPROXIMATION OF NAVIER–STOKES SYSTEM WITH COMPLEX RHEOLOGIES

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Abstract. In this work, we present a family of layer-averaged models for the Navier–Stokes equations. For its derivation, we consider a layerwise linear vertical profile for the horizontal velocity component. As a particular case, we also obtain layer-averaged models with the common layerwise constant approximation of the horizontal velocity. The approximation of the derivatives of the velocity components is set by following the theory of distributions to account for the discontinuities at the internal interfaces. Several models has been proposed, depending on the order of approximation of an asymptotic analysis respect to the shallowness parameter. Then, we obtain a hydrostatic model with vertical viscous effects, a hydrostatic model where the pressure depends on the stress tensor, and fully non-hydrostatic models, with a complex rheology. It is remarkable that the proposed models generalize plenty of previous models in the literature. Furthermore, all of them satisfy an exact dissipative energy balance. We also propose a model that is second-order accurate in the vertical direction thanks to a correction of the shear stress approximation. Finally, we show how effective the layerwise linear approach is to notably improve, with respect to the layerwise constant method, the approximation of the velocity profile for some geophysical flows. Namely, a Newtonian fluid and some complex viscoplastic (dry granular and Herschel–Bulkley) materials are considered.

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1. Introduction

Many efforts have been devoted to the study of geophysical flows, in particular, when they are involved in natural hazards. An advanced understanding of these flows is essential to develop efficient early warning systems against tsunamis, floods, storm surges, landslides, snow avalanches and volcanic eruptions among many others. However, there are still many open questions surrounding these phenomena, where three different aspects could be highlighted: the geophysical understanding of these flows and the definition of complex rheologies, the mathematical modelling through more sophisticated models including these rheologies, and the design of efficient numerical methods to approximate these complex models. Any advance in one of these fields, which are closely interconnected, is a valuable contribution to reach a more complete knowledge of natural flows.

Keywords and phrases. Layer-averaged systems, Navier–Stokes system, energy balance, dimensional analysis, complex rheology.

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This paper focuses on the second issue, that is, the development of sophisticated models including complex rheological terms. As starting point, it is common to consider the Navier–Stokes system [11] (or some others related systems as the Jackson's model [31] for two-phase flows). However, it is a known fact the prohibitive computational cost of approximating the solution of the 3D Navier–Stokes system. For this reason, a very active topic of research is the derivation of simplified models. A well-known strategy is to consider Shallow-Water (or Saint-Venant) type systems (see *e.g.* [29]). They are reached from two main ingredients: first, an asymptotic analysis in terms of the ratio between characteristic horizontal and vertical dimensions (called shallowness parameter ε), where vertical variations are supposed to be negligible when compared to horizontal ones; second, a depth-averaging of the 3D system, where it is necessary to set a vertical profile of the velocity. Actually, a constant vertical profile of the horizontal velocity component is assumed in most cases, leading to classical Shallow Water systems. Furthermore, as a consequence of the asymptotic analysis, viscous effects are commonly neglected and it yields to a hydrostatic definition of the pressure.

The simplifications made usually in shallow systems could be summarized as: (i) absence of viscous effects; (ii) a hydrostatic pressure; (iii) a horizontal velocity that is constant along the vertical direction. In order to overcome each of them, several strategies have been introduced in the literature.

Concerning the inclusion of viscous effects in Shallow-Water type models, let us mention some previous works. In [29], the viscous Shallow Water system, which includes horizontal viscous effects in the horizontal momentum conservation equation, was derived from an asymptotic analysis up to second order. A two-dimensional version of that model with capillary effects was introduced in [40]. Furthermore, the existence of global weak solutions for this model was proven in [6] where the so-called BD entropy was introduced. The model in [6] was generalized to the bilayer (stratified flow) case in [43], and the existence of weak solutions for this model was proven in [42]. In these works, in order to prove these results, authors needed to include capillarity or friction effects. A generalization of these results without neither capillarity nor friction terms was introduced in [41], where authors proved the stability of weak solutions of the barotropic compressible Navier–Stokes system. Notice that these results for the Saint-Venant system [29] are obtained from [41] as a particular case. In [47] the existence of global weak solutions for the compressible 3D Navier–Stokes system with degenerate viscosity was proven.

In previous works Newtonian fluids are considered. Nevertheless, viscous effects have been also included in shallow models for non-Newtonian fluids. For instance, in [30] authors introduced a viscous Shallow-Water type model for dry granular flows with a $\mu(I)$-viscosity, where they assumed a Bagnold vertical profile for the horizontal velocity. Other depth-averaged models for viscoplastic flows, as Bingham or Herschel–Bulkley fluids, including viscous effects, have been also presented (see [1, 22] among others).

The second important simplification we have remarked is the hydrostatic pressure that is commonly assumed in shallow flows. Many authors have studied the so-called dispersive models trying to go beyond this hydrostatic framework. When looking at the literature, two different families of dispersive systems are found: Boussinesq type and non-hydrostatic models. The main difference between these two families is the fact that Boussinesq models have as unknowns the total fluid depth and the averaged horizontal velocity, similarly to Saint-Venant systems (see [5,33,38,39,46] among many others), whereas non-hydrostatic models have extra unknowns related to the pressure (see *e.g.* [7,9,48]), leading to larger systems. Nevertheless, the main advantage of non-hydrostatic models with respect to Boussinesq type systems is that only first-order derivatives of the unknowns appear in the model, while high-order derivatives have to be discretized in Boussinesq models. Actually, many well-known Boussinesq type systems can be reformulated as non-hydrostatic models (see *e.g.* [7, 16]), which makes possible to design different numerical techniques to approximate them. All these models are compared in terms of their dispersion properties, as the dispersion relation, group velocity and linear shoaling, measuring the ability of the models to reproduce dispersive water waves of (very) high frequency. For non-Newtonian fluids, in [27] a shallow model for dry granular flows with a non-hydrostatic pressure is derived. This model took into account the vertical acceleration but neglected viscous effects in the pressure. Although it is not a fully non-hydrostatic approximation for granular flows, this model is able to recover some important results, as the parabolic shape of the front velocity in granular collapses.

The last point we previously highlighted is the lack of vertical resolution on the velocity profile surrounding depth-averaged models. The improvement of this vertical profile has been a very active research area in last years. There are several approaches. Some authors used information about the vertical structure in particular configurations to approximate some terms in their models, or to recover the vertical profile of velocity as a post-process. In the field of non-Newtonian fluids, in [3] the analytical solution for a uniform configuration is employed to get the averaged analytical velocity, and to deduce a lubrication model. The solution of uniform (or quasi-uniform) flows is also employed to improve the asymptotic expansion of the horizontal velocity. For instance, this strategy was followed in [22] to derive a shallow model. Other example is [10], where the influence of first-order corrections was studied through a formal asymptotic expansion of the velocity with respect to shallowness parameter ε in quasi-uniform flows. In [30], authors followed a similar strategy to approximate some terms in their model for dry granular flows. However, in previous works the models have not a vertical structure by themselves. Moreover, a vertical profile (for instance Bagnold type) is assumed for the velocity and it is not possible to recover other shapes of the velocity profile, or even changes on the velocity profile in transient problems.

Some other alternatives to recover the vertical structure of the fluid, without using a prescribed profile, consist on developing strategies based on approximating the vertical profile of the velocity. Let us remark here two of them. On the one hand, the so-called moment models, which were introduced for shallow flows in [35,36], allows us to recover the vertical profile of velocity through a polynomial approximation. These models have been used in several applications in a hydrostatic framework (see *e.g.* [25,34]). On the other hand, layer-averaged (or multilayer) models (see [2,21]), which are explained in the following paragraph, have been developed more widely for shallow flows, including its application to non-Newtonian fluids. Interestingly, both approaches can be written in a common framework, as shown in [26]. In that work, authors also proposed the combination of these methods to obtain multilayer-moment models, which are expected to be high-order accurate in the vertical direction, although it has been developed just in the hydrostatic framework for now.

Layer-averaged models consider a partition of the domain along the vertical direction in shallow layers. At the internal interfaces, which are not physical contrary to stratified flows, mass and momentum transference is allowed. Moreover, some unknowns can be discontinuous at these interfaces. In this approach, a preset vertical profile of the unknowns inside each layer is assumed to derive the model. It is worth mentioning that the global stability of weak solutions for the system in [2] with diffusive terms was proven in [14]. In the hydrostatic framework, several models with layerwise constant horizontal velocity have been presented, including also complex rheologies. For instance, see [18] for dry granular flows with the $\mu(I)$-rheology, [24] for two-phase (granular-fluid) flows with dilatancy effects, or [20] for Herschel–Bulkley fluids. In particular, in [18], the ability of layer-averaged models to change the shape of the vertical profile of velocity in transient problems depending on the flow regime was shown. This entailed an important improvement with respect to previous models that prescribed the vertical profile of the velocity. We must remark here that including rheological laws in the continuum solver is an essential aspect. For complex flows including a plasticity criterion, as the commented above, it is interesting to consider regularization methods (see [37,44]), mainly for two reasons: they allow us an easier writing of the model, and they lead to conceptually simpler numerical solvers, which have been shown to be efficient in different applications (see *e.g.* [18,20] among others).

Previous models only take into account some vertical viscous effects, whereas a layer-averaged hydrostatic model including all viscous terms was introduced in [8]. Let us remark that it is a very interesting work, which proposed a discretization of the stress tensor components that allows the model to satisfy an exact energy balance depending on the chosen rheology. Nevertheless, this model involves third-order derivatives making difficult its numerical approximation. Actually, numerical results has not been presented yet.

Very recently, a wide variety of non-hydrostatic layer-averaged models approximating Euler equations has been introduced. A family of models was proposed in [23], where they assumed that the horizontal velocity is layerwise constant. In that work, several models were presented depending on the degree of the polynomial approximation of the vertical velocity and the non-hydrostatic counterpart of the pressure. There, discontinuities could arise in the velocity field but not in the pressure, which is supposed to be continuous along the vertical direction. Furthermore, the dispersion relation of these layer-averaged models converges, when increasing the number of vertical layers, to the exact dispersion relation of Euler system, based on the Airy theory. This family of models were extended in [15] with an analogous procedure to the layerwise linear horizontal velocity case. Actually, it can be seen as a second-order approximation in the vertical direction of Euler equations. In that case, the analysis is more complicated, but it was an important improvement with respect to the previous work, especially when looking at the dispersion properties (dispersion relation, group velocity and linear shoaling). The results showed that, even with a very small number of vertical layers (1 or 2 layers), excellent dispersion properties are recovered. In that model, a comparison with previous models in [23] was also performed, showing how the layerwise linear horizontal velocity approach notably improved the results with respect to the layerwise constant one. All the presented models in the two previous works satisfy exact dissipative energy balances. Let us remark that these non-hydrostatic layer-averaged models did not consider viscous terms since they dealt with Euler equations.

In this work we focus on solving the three simplifications listed above simultaneously. Concretely, we derive non-hydrostatic layer-averaged models approximating the Navier–Stokes system, including all the viscous terms for a general stress tensor, with layerwise linear profile of both the horizontal velocity and the viscosity coefficient. As starting point, we consider the models introduced in [15] for Euler system. The inclusion of the stress tensor involves important difficulties. For instance, we use a definition of the derivatives of the velocity components accounting for the discontinuities at the internal interfaces, which allows us to prove an exact dissipative energy balance for the resulting models. The models in this work can be also presented as a hierarchy of models in terms of an asymptotic expansion with respect to the shallowness parameter (ε). Furthermore, the models here generalize in different senses (Euler to Navier–Stokes case, layerwise constant to linear approach, layerwise constant to linear shear stress, ...) many of previous models in the literature. We also give in this work a correction for the shear stress that allows the model to be a second-order discretization in the vertical direction of the Navier–Stokes system. It is illustrated with numerical results for some geophysical flows in simple configurations.

The paper is organised as follows: Section 2 is devoted to the initial system and notation. In Section 3 we detail the assumptions to approximate the variables, as well as the stress tensor. In particular Section 3.2 focuses on the case of a stress tensor proportional to the strain rate tensor, giving all the definitions that play a key role to get a dissipative energy balance. In Section 4, the non-hydrostatic layer-averaged models with layerwise linear horizontal velocity are derived, and the corresponding energy balances are proven. In Section 4.2 we present a second-order correction of the shear stress, which ensures the second-order accuracy in the approximation of the stress tensor. The compact writing of the proposed models is presented in Section 4.4. Section 5 is devoted to obtain and analyse a family of non-hydrostatic models from an asymptotic analysis, depending on the order of approximation on the shallowness parameter (ε). Concretely, in Section 5.2 we present a hydrostatic model with viscous dependent pressure, which is a generalization to the layerwise linear horizontal velocity case of the model introduced in [8]. A detailed summary of the models in this work, as well as its relation with previous models in the literature, is given in Section 5.4. Section 6 is devoted to show the accuracy of the layerwise linear approach to reproduce vertical profiles of velocity and shear stress for some uniform geophysical flows, including complex viscoplastic fluids. Finally, some conclusions are in Section 7.

2. Governing system and layer-averaged notation

We establish here the initial system and notation that we shall use along this work. The 2-dimensional space is considered here, then let $(x,z)\in\mathbb{R}^2$ be the space variables, with $\nabla=(\partial_x,\partial_z)$ the usual differential operators. We consider a constant density $\rho\in\mathbb{R}$ for an incompressible fluid, which flows within the domain

$$\Omega(t)=\left\{(x,z)\in\mathbb{R}^2 : b(x)<z<b(x)+H(t,x)\right\},$$

being $b(x)$ and $H(t,x)$ a bottom topography and the total depth of the fluid, respectively. $\boldsymbol{U}=(u,w)'\in\mathbb{R}^2$ denotes the vector velocity of the flow. Let us also denote by $\boldsymbol{g}=(g_x,g_z)$ an external force, which is usually defined by the gravitational acceleration. The dynamics of this flow is described by the Navier–Stokes system

$$\begin{cases}\partial_t\rho + \nabla\cdot(\rho\boldsymbol{U}) = 0,\\ \partial_t(\rho\boldsymbol{U}) + \nabla\cdot(\rho\boldsymbol{U}\otimes\boldsymbol{U}) \,-\, \nabla\cdot\boldsymbol{\sigma} = \rho\,\boldsymbol{g},\end{cases} \tag{1}$$

where $\boldsymbol{\sigma}$ is the total stress tensor defined by

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}$$

with p the total pressure, $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{\tau}$ the deviatoric tensor.

Following the approach of the so-called non-hydrostatic models (see *e.g.* [9]), the total pressure (p) is decomposed into hydrostatic and non-hydrostatic (q) contributions

$$p = \rho(-g_z(b + H - z) + q).$$

Finally, the definition of the deviatoric tensor ($\boldsymbol{\tau}$) depends on the rheological law describing the considered flow. Let us consider a general fluid (Newtonian of non-Newtonian), and define

$$\boldsymbol{\tau} = \begin{pmatrix}\tau_{xx} & \tau_{xz}\\ \tau_{zx} & \tau_{zz}\end{pmatrix}. \tag{2}$$

Notice that we have not removed the time derivative of the density despite of considering an incompressible flow. It is done for the purpose of clarity in the normal jump conditions associated to the layer-averaging approach.

One of the goals of this paper is to propose useful models to simulate geophysical flows, where local (or tilted) coordinates are commonly used (see [3, 4] among many others). Noticing that the Navier–Stokes equations are invariant under rotations, the only difference between these systems is the velocity direction and the definition of the external force $\boldsymbol{g}$.

In the case of local coordinates, the horizontal-vertical directions correspond to the downslope-normal directions to an inclined plane with constant slope $\tan\theta$, with $\theta \in (-\pi/2, \pi/2)$. For the sake of generality, let us consider

$$\boldsymbol{g} = (g_x, g_z)' = \begin{cases}(0, -g)' & \text{in Cartesian coordinates,}\\ (-g\,\sin\theta, -g\,\cos\theta)' & \text{in local coordinates,}\end{cases}$$

where $g \in \mathbb{R}$ is the gravity acceleration. We also introduce the following notation:

$$z_b(x) = b(x) - \frac{g_x}{|g_z|}\,x.$$

Note that the surface defined by z_b represents the bottom in Cartesian coordinates, where b is the local bottom, which is measured in the normal direction to the reference plane. In the case of Cartesian coordinates, we identify $z_b = b$.

Thus, (x, z) should be understood as the $x - z$ Cartesian system or the tilted reference system respect to the inclined plane. For the sake of simplicity, we shall write horizontal and vertical component of the velocity for (u, w), but it should be understood as downslope-normal components in the case of local coordinate systems.

As boundary conditions, we take zero atmospheric pressure and no tension at the free surface,

$$p_{|_{b+H}} = q_{|_{b+H}} = 0, \qquad \boldsymbol{\tau}_{|_{b+H}}\boldsymbol{n}^{b+H} = 0, \tag{3}$$

with $\boldsymbol{n}^{b+H}$ the downward unit normal vector, as well as the usual kinematic condition

$$\partial_t H + u_{|_{b+H}}\partial_X\,(b{+}H) - w_{|_{b+H}} = 0.$$

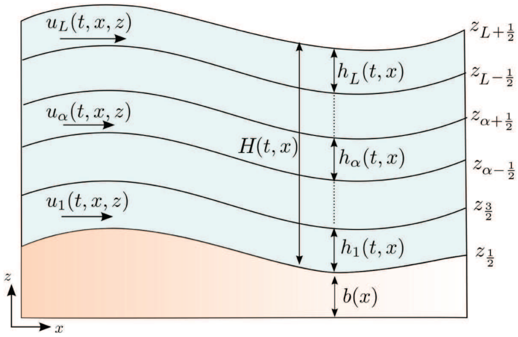

Figure 1. Sketch of the domain and its vertical partition.

At the bottom, the non-penetration condition is taken

$$u_{|_b}\partial_x b = w_{|_b}, \tag{4}$$

and for the sake of generality we shall consider a friction law

$$\boldsymbol{\sigma}\ \boldsymbol{n}^b - \left(\left(\boldsymbol{\sigma}\ \boldsymbol{n}^b\right)\cdot \boldsymbol{n}^b\right)\boldsymbol{n}^b = -\beta_0 \boldsymbol{U} - \beta_1\frac{\boldsymbol{U}}{|\boldsymbol{U}|}, \tag{5}$$

being $\boldsymbol{n}^b = (\partial_x b, -1)/\sqrt{1+(\partial_x b)^2}$ the downward unit normal vector at the bottom and β_0, β_1 friction coefficients. Notice that previous friction condition encompasses different boundary conditions depending on the considered flow (or material). For instance, in the case of Newtonian fluids, we set $\beta_1 = 0$ and $\beta_0 = \beta_0(|\boldsymbol{U}|)$ a constant or variable coefficient.

System (1) will be discretized in the framework of the layer-averaged approach introduced in [21]. To this aim, we consider a subdivision in the vertical direction of the domain into $L \in \mathbb{N}$ shallow layers Ω_α, whose heights are h_α for $\alpha = 1,\dots,L$ (see Fig. 1). Let $\mathcal{L}_{\alpha+1/2}$ be the interface separating the layers $\Omega_\alpha, \Omega_{\alpha+1}$, that is defined by the equation $z = z_{\alpha+1/2}$. Then $z_{\alpha+1/2} = b + \sum_{\beta=1}^{\alpha} h_\beta$ and

$$\Omega_\alpha(t) = \left\{(x,z)\in\mathbb{R}^2 : z_{\alpha-1/2} < z < z_{\alpha+1/2}\right\},$$

being $z_{1/2}$ and $z_{L+1/2}$ the bottom and free surface, respectively. Then, the total height of the fluid is $H = \sum_{\beta=1}^{L} h_\beta$, and it holds that $h_\alpha = z_{\alpha+1/2} - z_{\alpha-1/2}$. Moreover, the midpoint of each layer Ω_α is $z_\alpha = z_{\alpha-1/2} + h_\alpha/2$. Finally, a vertical mesh is defined through the coefficients $(\ell_\alpha)_{\{\alpha\in 1,\dots,L\}}$ satisfying

$$h_\alpha = \ell_\alpha H, \quad \text{with} \quad \ell_\alpha\in[0,1] \quad \text{and} \quad \sum_{\alpha=1}^{L}\ell_\alpha = 1.$$

Let us now fix the same notation as in [15] for an arbitrary function $f(t,x,z)$. We denote by $f^{\pm}_{\alpha+1/2}$ its approximation at the interface $\mathcal{L}_{\alpha+1/2}$, being

$$f^-_{\alpha+1/2} = \lim_{\substack{z\to z_{\alpha+1/2}\\ z<z_{\alpha+1/2}}} f_{|\Omega_\alpha}, \qquad f^+_{\alpha+1/2} = \lim_{\substack{z\to z_{\alpha+1/2}\\ z>z_{\alpha+1/2}}} f_{|\Omega_{\alpha+1}}.$$

We write $f_{\alpha+1/2}$ if both limits match and f is a continuous function. Now, the average over the layer Ω_α is denoted by $\overline{f}_\alpha$ and the linear average by $\widehat{f}_\alpha$, being

$$\overline{f}_\alpha(t,x) = \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha-1/2}} f(t,x,z)\,\mathrm{d}z, \qquad \text{and} \qquad \widehat{f}_\alpha = \frac{f^+_{\alpha-1/2} + f^-_{\alpha+1/2}}{2}.$$

Notice that $\overline{f}_\alpha = \widehat{f}_\alpha$ holds when f is linear (or constant) in the vertical variable z in the layer Ω_α. It is also useful to define the variation through the layer

$$(\delta f)_\alpha = f^-_{\alpha+1/2} - f^+_{\alpha-1/2}. \tag{6a}$$

Concerning the approximations at the interface $\mathcal{L}_{\alpha+1/2}$, we denote by $\widetilde{f}_{\alpha+1/2}$ and $[f]_{\alpha+1/2}$ the average value at this interfaces and the jump across them, respectively, defined by

$$\widetilde{f}_{\alpha+1/2} = \frac{f^+_{\alpha+1/2} + f^-_{\alpha+1/2}}{2}, \qquad \text{and} \qquad [f]_{\alpha+1/2} = f^+_{\alpha+1/2} - f^-_{\alpha+1/2}. \tag{6b}$$

3. Layerwise approximation: velocity, pressure and stress tensor closure

In this section, we detail the vertical profile for the variables in the layer-averaged framework, as in [15]. We later focus on the stress tensor approximation, also accounting for the possible discontinuous profile of the velocity.

Let us denote by

$$\boldsymbol{U}_\alpha := \boldsymbol{U}_{|\Omega_\alpha} := (u_\alpha, w_\alpha)',$$

the velocity in the layer Ω_α, where u_α and w_α are the horizontal and vertical components of the velocity. Then, we assume a linear profile in z for the horizontal velocity within each layer. That is, the horizontal velocity is layerwise linear ($u_\alpha \in \mathbb{P}_1$):

$$u_\alpha(z) = \overline{u}_\alpha + \lambda_\alpha\big(z - z_\alpha\big), \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}], \tag{7}$$

being $\overline{u}_\alpha$ the averaged velocity and λ_α its slope. Consequently, the limit values at the interfaces $u^\mp_{\alpha\pm1/2}$ are given by

$$u^-_{\alpha+1/2} = \overline{u}_\alpha + \frac{h_\alpha\lambda_\alpha}{2}, \qquad u^+_{\alpha-1/2} = \overline{u}_\alpha - \frac{h_\alpha\lambda_\alpha}{2}. \tag{8}$$

For the vertical velocity, looking at the incompressibility condition, we consider a layerwise parabolic profile ($w_\alpha \in \mathbb{P}_2$):

$$w_\alpha(z) = w_\alpha(z_\alpha) + \varphi_\alpha(z - z_\alpha) + \frac{\psi_\alpha}{2}(z - z_\alpha)^2, \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]. \tag{9}$$

Previous equation can be written also as

$$w_\alpha(z) = \overline{w}_\alpha + \varphi_\alpha(z - z_\alpha) + \frac{\psi_\alpha}{2}\left((z - z_\alpha)^2 - \frac{h_\alpha^2}{12}\right), \tag{10}$$

for $\alpha = 1, \dots, L$, using the averaged vertical velocity $\overline{w}_\alpha = w_\alpha(z_\alpha) + h_\alpha^2\psi_\alpha/24$. The limit values at the interfaces are then

$$w^-_{\alpha+1/2} = \overline{w}_\alpha + \frac{h_\alpha\varphi_\alpha}{2} + \frac{h_\alpha^2\psi_\alpha}{12}, \qquad w^+_{\alpha-1/2} = \overline{w}_\alpha - \frac{h_\alpha\varphi_\alpha}{2} + \frac{h_\alpha^2\psi_\alpha}{12}. \tag{11}$$

The variables $\varphi_\alpha, \psi_\alpha$ defining the vertical profile (10) can be related to the variables in the horizontal velocity (7) by means of the incompressibility condition (see [15] for details). Concretely, we obtain the constraints

$$\begin{cases} \varphi_\alpha = -\partial_x\overline{u}_\alpha + \lambda_\alpha\partial_x z_\alpha, \\ \psi_\alpha = -\partial_x\lambda_\alpha, \end{cases} \tag{12}$$

for $\alpha = 1, \dots, L$. Notice that both the horizontal and vertical velocities could be discontinuous across the interfaces $\mathcal{L}_{\alpha+1/2}$.

Concerning the non-hydrostatic pressure q, it is a layerwise cubic function ($q_\alpha \in \mathbb{P}_3$) accordingly to the vertical momentum equation. Moreover, it is assumed to be continuous across the interfaces ($q_\alpha(z_{\alpha\pm1/2}) = q_{\alpha\pm1/2}$). Considering the variable π_α satisfying

$$\partial_z q_\alpha(z_\alpha) = \frac{\pi_\alpha}{h_\alpha},$$

and using the notation in (6), the vertical profile of non-hydrostatic pressure is

$$q_\alpha(z) = \frac{3\overline{q}_\alpha - \widehat{q}_\alpha}{2} + \pi_\alpha \frac{z - z_\alpha}{h_\alpha} + 6(\widehat{q}_\alpha - \overline{q}_\alpha)\frac{(z - z_\alpha)^2}{h_\alpha^2} + 4\big((\delta q)_\alpha - \pi_\alpha\big)\frac{(z - z_\alpha)^3}{h_\alpha^3},$$

for $z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]$. Notice that here we assume to be far from dry areas. In these situations $q_\alpha = 0$ is expected, although it is not studied here. It will be addressed in the future when studying the numerical approximation of these models.

The main novelty with respect to previous work [15] is the fact of dealing with the Navier–Stokes system instead of Euler equations. More concretely, we include the viscous terms. So, we need to properly approximate the stress tensor components. To this aim, we also assume a polynomial approximation, where the coefficient must be defined in terms of the considered rheology, in order to approximate up to second-order the viscous terms appearing in the Navier–Stokes system ($\partial_x \tau_{xx}$, $\partial_z \tau_{xz}$, $\partial_x \tau_{xz}$, $\partial_z \tau_{zz}$). Let us consider the following definition of the layerwise stress tensor components,

$$\tau_{ij,\alpha}(z) = \overline{\tau}_{ij,\alpha} + \zeta_{ij,\alpha}(z - z_\alpha) + \xi_{ij,\alpha}\left(\frac{(z - z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right) + \varkappa_{ij,\alpha}\left(\frac{(z - z_\alpha)^3}{3} - \frac{h_\alpha^2}{20}(z - z_\alpha)\right), \tag{13}$$

where $i, j \in \{x, z\}$ and we assume $\tau_{xz,\alpha} = \tau_{zx,\alpha}$. Let us remark that the layer-averaged approximation developed in this paper remains valid for any application in which the deviatoric tensor is properly approximated by (13), as long as others extra unknowns and equations do not appear in the model (for instance, some turbulence models). Actually, our motivation to take these profiles lies in approximating a stress tensor defined as the product of the strain rate tensor and the kinematic viscosity (see Sect. 3.2). Thus, taking into account the hypothesis for u_α, w_α and by considering a linear approximation of the kinematic viscosity, we suppose that the stress tensor components are polynomials of degree $d \leq 3$.

In next section, we give some important relations for the stress tensor concerning to the normal jump conditions associated to the layerwise approach. In particular, the viscous terms at the interfaces $\mathcal{L}_{\alpha+1/2}$ and boundary conditions for the stress tensor are detailed.

3.1. Jump conditions and closure of the momentum transference terms

We focus now on the jump conditions associated to the mass and the momentum equations (1). We are looking for a particular weak solution $(\rho, \boldsymbol{U}, p)$ of (1), which must be a regular solution within each layer Ω_α, and satisfy the normal jump conditions across the internal interfaces $\mathcal{L}_{\alpha+1/2}$.

As usual, from the mass conservation equation we have

$$[(\rho;\ \rho\boldsymbol{U})]_{|\mathcal{L}_{\alpha+1/2}} \cdot \ \big(\partial_t z_{\alpha+1/2}, \partial_x z_{\alpha+1/2}, -1\big) = 0,$$

which gives the definition of the mass transfer terms $\Gamma_{\alpha+1/2}$ at the interface $\mathcal{L}_{\alpha+1/2}$:

$$\Gamma_{\alpha+1/2} := \Gamma^-_{\alpha+1/2} = \Gamma^+_{\alpha+1/2}, \qquad \text{with} \qquad \Gamma^\pm_{\alpha+1/2} = -\Big(\partial_t z_{\alpha+1/2} + u^\pm_{\alpha+1/2}\partial_x z_{\alpha+1/2} - w^\pm_{\alpha+1/2}\Big),$$

for $\alpha = 1, \ldots, L-1$. For the boundary cases $\Gamma_{1/2}$ and $\Gamma_{L+1/2}$, which account for the mass transfer with the bottom and free surface, respectively, we set them as zero.

From previous equation, an expression for the evolution of the layer midpoint can be deduced. Concretely,

$$\partial_t z_\alpha + \overline{u}_\alpha \partial_x z_\alpha - \widehat{w}_\alpha = -\frac{h_\alpha}{4}\lambda_\alpha \partial_x h_\alpha - \frac{\Gamma_{\alpha-1/2} + \Gamma_{\alpha+1/2}}{2}.$$

This equation is useful for both the layer-averaging procedure and the proof of the energy balance.

For the momentum conservation equation, we have

$$[(\rho \boldsymbol{U};\ \rho \boldsymbol{U}\otimes \boldsymbol{U} - \boldsymbol{\sigma})]_{|\mathcal{L}_{\alpha+1/2}} \cdot \big(\partial_t z_{\alpha+1/2}, \partial_x z_{\alpha+1/2}, -1\big) = 0.$$

By assuming that the dynamic pressure is continuous at the interface, previous equation is written in terms of the jump of the deviatoric tensor $\boldsymbol{\tau}$ as

$$[\boldsymbol{\tau}]_{|\mathcal{L}_{\alpha+1/2}} \cdot \big(\partial_x z_{\alpha+1/2}, -1\big) = -\rho\, \Gamma_{\alpha+1/2}\, [\boldsymbol{U}]_{|\mathcal{L}_{\alpha+1/2}}. \tag{14}$$

Let us define

$$K^{\pm}_{\alpha+1/2} = \frac{1}{\rho}\Big(\tau^{\pm}_{xx,\alpha+1/2}\, \partial_x z_{\alpha+1/2} - \tau^{\pm}_{xz,\alpha+1/2}\Big),$$
$$K^{\pm}_{w,\alpha+1/2} = \frac{1}{\rho}\Big(\tau^{\pm}_{xz,\alpha+1/2}\, \partial_x z_{\alpha+1/2} - \tau^{\pm}_{zz,\alpha+1/2}\Big).$$

Then, condition (14) is written by components as follows:

$$K^{+}_{\alpha+1/2} - K^{-}_{\alpha+1/2} = -\Gamma_{\alpha+1/2}\Big(u^{+}_{\alpha+1/2} - u^{-}_{\alpha+1/2}\Big),$$
$$K^{+}_{w,\alpha+1/2} - K^{-}_{w,\alpha+1/2} = -\Gamma_{\alpha+1/2}\Big(w^{+}_{\alpha+1/2} - w^{-}_{\alpha+1/2}\Big). \tag{15}$$

Moreover, following [21], it is necessary to set a closure condition. We set

$$\frac{K^{+}_{\alpha+1/2} + K^{-}_{\alpha+1/2}}{2} = K_{\alpha+1/2}, \qquad \frac{K^{+}_{w,\alpha+1/2} + K^{-}_{w,\alpha+1/2}}{2} = K_{w,\alpha+1/2} \tag{16}$$

where $K_{\alpha+1/2}$ and $K_{w,\alpha+1/2}$ are approximations of

$$\frac{1}{\rho}\big(\tau_{xx}\, \partial_x z_{\alpha+1/2} - \tau_{xz}\big)_{|\mathcal{L}_{\alpha+1/2}}, \qquad \text{and} \qquad \frac{1}{\rho}\big(\tau_{xz,\alpha}\, \partial_x z_{\alpha+1/2} - \tau_{zz}\big)_{|\mathcal{L}_{\alpha+1/2}},$$

respectively. Let us notice that other convex combination could be used in principle in (16).

Therefore, by using the jump conditions (15) and closure (16), we obtain

$$K^{\pm}_{\alpha+1/2} = K_{\alpha+1/2} \mp \frac{1}{2}\Gamma_{\alpha+1/2}\Big(u^{+}_{\alpha+1/2} - u^{-}_{\alpha+1/2}\Big), \tag{17a}$$
$$K^{\pm}_{w,\alpha+1/2} = K_{w,\alpha+1/2} \mp \frac{1}{2}\Gamma_{\alpha+1/2}\Big(w^{+}_{\alpha+1/2} - w^{-}_{\alpha+1/2}\Big). \tag{17b}$$

Then, the limits of the deviatoric tensor at the interfaces are written in terms of the averaged values and the mass transference terms. This definition of the momentum transference terms $K_{\alpha+1/2}$ deserves special attention in order to ensure that the model satisfies a dissipative energy balance. Following [8], we consider the approximation

$$K_{\alpha+1/2} = \frac{1}{\rho}\left(\frac{\overline{\tau}_{xx,\alpha} + \overline{\tau}_{xx,\alpha+1}}{2}\, \partial_x z_{\alpha+1/2} - \frac{\overline{\tau}_{xz,\alpha} + \overline{\tau}_{xz,\alpha+1}}{2}\right),$$
$$K_{w,\alpha+1/2} = \frac{1}{\rho}\left(\frac{\overline{\tau}_{xz,\alpha} + \overline{\tau}_{xz,\alpha+1}}{2}\, \partial_x z_{\alpha+1/2} - \frac{\overline{\tau}_{zz,\alpha} + \overline{\tau}_{zz,\alpha+1}}{2}\right), \tag{18a}$$

for $\alpha = 1, \ldots, L-1$, where $\overline{\tau}_{xx,\alpha}, \overline{\tau}_{xz,\alpha}, \overline{\tau}_{zz,\alpha}$ for $\alpha = 1, \ldots, L$ are the averaged stress tensor components in (13), which must be defined appropriately.

The values of K, K_w at the bottom and free surface are given by the boundary conditions (3) and (5). Concretely, we consider

$$K_{1/2} = \frac{1}{\rho}\left(\tau_{xx|_b}\partial_x b - \tau_{xz|_b}\right), \qquad K_{L+1/2} = \frac{1}{\rho}\left(\tau_{xx|_{b+H}}\partial_x (b+H) - \tau_{xz|_{b+H}}\right),$$
$$K_{w,1/2} = \frac{1}{\rho}\left(\tau_{xz|_b}\partial_x b - \tau_{zz|_b}\right), \qquad K_{w,L+1/2} = \frac{1}{\rho}\left(\tau_{xz|_{b+H}}\partial_x (b+H) - \tau_{zz|_{b+H}}\right).$$

From (3) we deduce that

$$K_{L+1/2} = K_{w,L+1/2} = 0, \tag{18b}$$

and (5) gives two conditions

$$\begin{pmatrix} K_{1/2} + K_{w,1/2}\partial_x b \\ K_{1/2}\partial_x b + K_{w,1/2}(\partial_x b)^2 \end{pmatrix} = -\frac{1}{\rho}\left(1 + (\partial_x b)^2\right)^{3/2}\beta\left(\left|\boldsymbol{U}^+_{1/2}\right|\right)\boldsymbol{U}^+_{1/2},$$

where

$$\beta\left(\left|\boldsymbol{U}^+_{1/2}\right|\right) = \beta_0 + \frac{\beta_1}{\left|\boldsymbol{U}^+_{1/2}\right|}, \quad \text{with} \quad \boldsymbol{U}^+_{1/2} = \left(u^+_{1/2}, w^+_{1/2}\right)' = \left(u^+_{1/2}, u^+_{1/2}\partial_x b\right)'$$

thanks to the non-penetration condition (4), and $u^+_{1/2} = \overline{u}_1 - \frac{h_1\lambda_1}{2}$ the horizontal velocity at the bottom. Note that previous equation gives a linear system whose unknowns are $K_{1/2}, K_{w,1/2}$. Actually, the second equation equals the first one multiplied by $\partial_x b$. Therefore, this relation is mandatory also for the right-hand side term in order to get a consistent (underdetermined) system. Hopefully, this holds thanks to the non-penetration condition (4). Otherwise, is not possible to define such a friction condition.

Then, to define $K_{1/2}, K_{w,1/2}$, we consider any expression satisfying

$$K_{1/2} + K_{w,1/2}\partial_x b = -\frac{1}{\rho}\left(1 + (\partial_x b)^2\right)^{3/2}\beta\left(\boldsymbol{U}^+_{1/2}\right)u^+_{1/2}.$$

For instance, without loss of generality, we could define

$$K_{1/2} = -\frac{1}{\rho}\sqrt{1 + (\partial_x b)^2}\,\beta\left(\boldsymbol{U}^+_{1/2}\right)u^+_{1/2}, \qquad \text{and} \qquad K_{w,1/2} = \partial_x b\, K_{1/2}. \tag{18c}$$

In next section, we develop all the details for the common case of having the stress tensor as the product of the strain rate tensor and the kinematic viscosity.

3.2. Stress tensor proportional to the strain rate tensor

Denoting the kinematic viscosity coefficient by ν, which could be variable, we consider the stress tensor (2) given by

$$\boldsymbol{\tau} = \rho\nu D(\boldsymbol{U}) \qquad \text{where} \qquad D(\boldsymbol{U}) = \frac{1}{2}(\nabla\boldsymbol{U} + (\nabla\boldsymbol{U})').$$

Therefore, its components are

$$\tau_{xx} = \rho\nu\partial_x u, \quad \tau_{xz} = \tau_{zx} = \rho\frac{\nu}{2}(\partial_x w + \partial_z u), \quad \tau_{zz} = \rho\nu\partial_z w.$$

Let us recall that we have set here u, w as layerwise linear and parabolic functions, respectively (see (7) and (9)). In addition, a goal of this work is to propose models for geophysical flows, which are represented by appropriate rheological laws. These laws can be defined through variable viscosity coefficients, which could depend, for instance, on the velocity and pressure. Therefore, the viscosity is also a function that must be approximated in the layer-averaged framework. In [18], a layerwise constant viscosity is assumed for viscoplastic granular flows. To the aim of improving the approximation of the viscosity for complex flows and reaching second-order accuracy of the stress tensor components, in this work we go further and assume a linear approximation of the viscosity within each layer. Thus, it is defined by

$$\nu_{ij,\alpha}(z) = \nu^0_{ij,\alpha} + \nu^1_{ij,\alpha}(z - z_\alpha), \qquad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}], \qquad i,j \in \{x,z\}, \tag{19}$$

and $\alpha = 1,\dots,L$, such that $\nu_{ij,\alpha} \geq 0$. In Section 6 we will see that this approach is appropriate for several rheologies. Note that we are considering a different viscosity coefficient $\nu_{ij,\alpha}$ for each component of the deviatoric stress tensor, thus making broader the range of applicability of the proposed models. For instance, it is useful in the case of turbulent flows or, in general, when having different viscosity coefficients along the horizontal and vertical directions (see *e.g.* [13]). Notice also that in the case of Newtonian fluids we have a constant profile, then $\nu_{ij,\alpha} = \nu^0_{ij,\alpha} = \nu$ being ν the constant kinematic viscosity of the fluid.

A key point in the definition of the approximation of the stress tensor components is the approximation of $\partial_x u$, $\partial_x w$, $\partial_z u$ and $\partial_z w$ accounting for the possible discontinuities at the interfaces $\mathcal{L}_{\alpha+1/2}$ of the velocity components u and w.

Let us remind the reader that, for a fixed time $t > 0$, for any vector function $\boldsymbol{F}(t,x,z) \in \Omega \subset \mathbb{R}^2$ being a regular solution within each layer Ω_α, for $\alpha = 1,\dots,L$, with possible discontinuities at the internal interfaces $\mathcal{L}_{\alpha+1/2}$, for $\alpha = 1,\dots,L-1$, we can define the divergence $[div_{(x,z)}\boldsymbol{F}(t,\cdot,\cdot)]$ in the sense of distributions

$$\begin{aligned}\big\langle \big[div_{(x,z)}\boldsymbol{F}(t,\cdot,\cdot)\big], \phi\big\rangle = & \int_\Omega div_{(x,z)}\boldsymbol{F}(t,x,z)\phi(x,z)\,\mathrm{d}x\,\mathrm{d}z \\ & + \int_{I_\Omega}\sum_{\alpha=1}^{L-1}\Big(\boldsymbol{F}^+_{\alpha+1/2} - \boldsymbol{F}^-_{\alpha+1/2}\Big)\cdot\begin{pmatrix}-\partial_x z_{\alpha+1/2}\\ 1\end{pmatrix}\phi\big(x, z_{\alpha+1/2}\big)\,\mathrm{d}x, \quad \forall\phi \in \mathcal{D}(\Omega),\end{aligned}$$

where $\boldsymbol{F}^\pm_{\alpha+1/2}(x)$ are the upper and lower limits of $\boldsymbol{F}(t,x,z)$ when z tends to $z_{\alpha+1/2}$, respectively. $\mathcal{D}(\Omega)$ is the set of functions of class $C^\infty(\Omega)$ with compact support, I_Ω the projection of Ω over $\mathbb{R}$, and the divergence operator appearing in the double integral has to be understood in the pointwise sense.

Previous equation motivates the following definitions of the derivatives of the velocity components:

$$\begin{aligned}\overline{[\partial_z u]}_\alpha &= \overline{\partial_z(u_\alpha(z))}_\alpha + \frac{1}{h_\alpha}\frac{[u]_{\alpha+1/2} + [u]_{\alpha-1/2}}{2} = \lambda_\alpha + \frac{1}{h_\alpha}\frac{[u]_{\alpha+1/2} + [u]_{\alpha-1/2}}{2},\\ \overline{[\partial_z w]}_\alpha &= \overline{\partial_z(w_\alpha(z))}_\alpha + \frac{1}{h_\alpha}\frac{[w]_{\alpha+1/2} + [w]_{\alpha-1/2}}{2} = \varphi_\alpha + \frac{1}{h_\alpha}\frac{[w]_{\alpha+1/2} + [w]_{\alpha-1/2}}{2},\\ \overline{[\partial_x u]}_\alpha &= \overline{\partial_x(u_\alpha(z))}_\alpha - \frac{1}{h_\alpha}\frac{[u]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [u]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2}\\ &= \partial_x\overline{u}_\alpha - \lambda_\alpha\partial_x z_\alpha - \frac{1}{h_\alpha}\frac{[u]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [u]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2},\\ \overline{[\partial_x w]}_\alpha &= \overline{\partial_x(w_\alpha(z))}_\alpha - \frac{1}{h_\alpha}\frac{[w]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [w]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2}\\ &= \partial_x\overline{w}_\alpha - \varphi_\alpha\partial_x z_\alpha - \frac{h_\alpha\psi_\alpha}{12}\partial_x h_\alpha - \frac{1}{h_\alpha}\frac{[w]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [w]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2},\end{aligned} \tag{20}$$

for $\alpha = 1,\dots,L$, where we take the jumps $[u]_{1/2}$, $[w]_{1/2}$, $[u]_{L+1/2}$ and $[w]_{L+1/2}$ as zero. Let us remark that these definitions can be seen as a partition of $\langle div_{(x,z)}\boldsymbol{F}, \mathbf{1}_\Omega\rangle$ for $\boldsymbol{F} = (0,u)'$, $\boldsymbol{F} = (0,w)'$, $\boldsymbol{F} = (u,0)$, and $\boldsymbol{F} = (w,0)$, respectively, satisfying then

$$\langle[\partial_z u(t,\cdot,\cdot)], \mathbf{1}_\Omega\rangle = \sum_{\alpha=1}^{L}\int_{I_\Omega} h_\alpha\overline{[\partial_z u]}_\alpha\,\mathrm{d}x, \qquad \langle[\partial_z w(t,\cdot,\cdot)], \mathbf{1}_\Omega\rangle = \sum_{\alpha=1}^{L}\int_{I_\Omega} h_\alpha\overline{[\partial_z w]}_\alpha\,\mathrm{d}x,$$

$$\langle[\partial_x u(t,\cdot,\cdot)],\mathbf{1}_\Omega\rangle = \sum_{\alpha=1}^{L}\int_{I_\Omega} h_\alpha\overline{[\partial_x u]}_\alpha\,\mathrm{d}x, \qquad \langle[\partial_x w(t,\cdot,\cdot)],\mathbf{1}_\Omega\rangle = \sum_{\alpha=1}^{L}\int_{I_\Omega} h_\alpha\overline{[\partial_x w]}_\alpha\,\mathrm{d}x.$$

Remark 1. Following [45], the contribution of the jumps at the interfaces $\mathcal{L}_{\alpha+1/2}$ in (20) has been equally distributed between layers Ω_α and $\Omega_{\alpha+1}$. In principle, any convex combination could be used, and in that case condition (16) should be accordingly modified.

Now, we specify the expression of the variables in the approximations of the stress tensor components at each layer (13), for which we use the definitions in (20). Taking into account the linear profile of the viscosity (19), τ_{xx} is approximated at each layer by $\tau_{xx,\alpha}$, where we set

$$\tau_{xx,\alpha}(z) = \left(\nu^0_{xx,\alpha} + \nu^1_{xx,\alpha}(z-z_\alpha)\right)\Big(\overline{[\partial_x u]}_\alpha + \partial_x\lambda_\alpha(z-z_\alpha)\Big). \tag{21}$$

Note that it can be rewritten under the form (13) with the following definition of the components:

$$\begin{aligned}
\overline{\tau}_{xx,\alpha} &= \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\tau_{xx,\alpha}(z)\,\mathrm{d}z = \nu^0_{xx,\alpha}\overline{[\partial_x u]}_\alpha + \nu^1_{xx,\alpha}\frac{h_\alpha^2}{12}\partial_x\lambda_\alpha,\\
\zeta_{xx,\alpha} &= \nu^1_{xx,\alpha}\overline{[\partial_x u]}_\alpha + \nu^0_{xx,\alpha}\partial_x\lambda_\alpha,\\
\xi_{xx,\alpha} &= 2\nu^1_{xx,\alpha}\partial_x\lambda_\alpha,\\
\varkappa_{xx,\alpha} &= 0.
\end{aligned} \tag{22}$$

For the approximation of τ_{xz} at each layer, we consider

$$\begin{aligned}
\tau_{xz,\alpha}(z) = &\frac{\nu^0_{xz,\alpha} + \nu^1_{xz,\alpha}(z-z_\alpha)}{2}\\
&\times\left(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]}_\alpha + (\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha)(z-z_\alpha) + \partial_x\psi_\alpha\left(\frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right)\right).
\end{aligned} \tag{23}$$

Then, $\tau_{xz,\alpha}$ can be defined by (13), where

$$\begin{aligned}
\overline{\tau}_{xz,\alpha} &= \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\tau_{xz,\alpha}(z)\,\mathrm{d}z = \frac{\nu^0_{xz,\alpha}}{2}\left(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]}_\alpha\right) + \frac{\nu^1_{xz,\alpha}h_\alpha^2}{24}(\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha),\\
\zeta_{xz,\alpha} &= \frac{\nu^1_{xz,\alpha}}{2}\left(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]}_\alpha + \frac{h_\alpha^2}{30}\partial_x\psi_\alpha\right) + \frac{\nu^0_{xz,\alpha}}{2}(\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha),\\
\xi_{xz,\alpha} &= \frac{\nu^0_{xz,\alpha}}{2}\partial_x\psi_\alpha + \nu^1_{xz,\alpha}(\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha),\\
\varkappa_{xz,\alpha} &= \frac{3}{4}\nu^1_{xz,\alpha}\partial_x\psi_\alpha.
\end{aligned} \tag{24}$$

Finally, we take the approximation of τ_{zz} at each layer given by

$$\tau_{zz,\alpha}(z) = \left(\nu^0_{zz,\alpha} + \nu^1_{zz,\alpha}(z-z_\alpha)\right)\Big(\overline{[\partial_z w]}_\alpha + \psi_\alpha(z-z_\alpha)\Big). \tag{25}$$

In this case, it is written under form (13) through the components

$$\begin{aligned}
\overline{\tau}_{zz,\alpha} &= \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\tau_{zz,\alpha}(z)\,\mathrm{d}z = \nu^0_{zz,\alpha}\overline{[\partial_z w]}_\alpha + \nu^1_{zz,\alpha}\frac{h_\alpha^2}{12}\psi_\alpha,\\
\zeta_{zz,\alpha} &= \nu^1_{zz,\alpha}\overline{[\partial_z w]}_\alpha + \nu^0_{zz,\alpha}\psi_\alpha,\\
\xi_{zz,\alpha} &= 2\nu^1_{zz,\alpha}\psi_\alpha,\\
\varkappa_{zz,\alpha} &= 0.
\end{aligned} \tag{26}$$

In the next section we derive the non-hydrostatic layer-averaged Navier–Stokes system.

4. Layer-averaged systems with linear horizontal velocity

As in our previous work [15], we can derive a family of models depending of the degree of approximation of the vertical velocity. We first consider the case of a layerwise parabolic vertical velocity ($w_\alpha \in \mathbb{P}_2$), where the degree of approximation of the vertical velocity is given by the incompressibility condition. Consequently to the vertical momentum conservation equation, the non-hydrostatic pressure is a third-order polynomial ($q_\alpha \in \mathbb{P}_3$) in that case. This model is denoted by LIN-NH$_2$-STRESS. However, as showed in [15] this can be relaxed to the case of layerwise linear vertical velocity ($w_\alpha \in \mathbb{P}_1$ and $q_\alpha \in \mathbb{P}_2$) keeping excellent dispersion properties (dispersion relation, group velocity and linear shoaling). We will obtain the corresponding simplified model, denoted by LIN-NH$_1$-STRESS, as a particular case of the LIN-NH$_2$-STRESS model.

In next, we focus first on the more complete model (LIN-NH$_2$-STRESS), as well as a second-order correction for the shear stress. We later present the simplified model with linear vertical velocity.

4.1. Fully non-hydrostatic model for a general stress tensor: LIN-NH$_2$-STRESS

In order to obtain the models, we perform here the usual layer-averaging procedure. From the mass conservation law, we recover the usual layerwise mass conservation equation

$$\partial_t h_\alpha + \partial_x(h_\alpha u_\alpha) = -\Gamma_{\alpha+1/2} + \Gamma_{\alpha-1/2}, \qquad \alpha = 1,\dots,L.$$

Combining previous equations, namely summing up from 1 to L, we get an equation for the total height

$$\partial_t H + \partial_x\big(H\overline{\overline{u}}\big) = 0, \quad \text{with} \quad \overline{\overline{u}} = \sum_{\alpha=1}^{L} \ell_\alpha \overline{u}_\alpha,$$

as well as an explicit expression for the mass transference term at the interfaces $\mathcal{L}_{\alpha+1/2}$ in terms of the velocities and the fluid depth

$$\Gamma_{\alpha+1/2} = \sum_{\beta=\alpha+1}^{L} \ell_\beta \partial_x\big(H\big(\overline{u}_\beta - \overline{\overline{u}}\big)\big), \qquad \text{for } \alpha = 1,\dots,L-1.$$

Thus, the mass transference are no more unknowns.

One can observe that we have 8L+1 unknowns. Concretely, the total height (H), 5L variables for the velocity field ($\{\overline{u}_\alpha, \lambda_\alpha, \overline{w}_\alpha, \varphi_\alpha, \psi_\alpha\}$ for $\alpha = 1,\dots,L$), and 3L extra unknowns for the pressure field ($\{\overline{q}_\alpha, q_{\alpha-1/2}, \pi_\alpha\}$ for $\alpha = 1,\dots,L$). First, the evolution equations for the conserved variables coming from the velocity field ($\{h_\alpha\overline{u}_\alpha, h_\alpha^2\lambda_\alpha, h_\alpha\overline{w}_\alpha, h_\alpha^2\varphi_\alpha, h_\alpha^3\psi_\alpha\}$) are obtained from the momentum conservation laws. Concretely, we need an extra equation for the horizontal velocity and two of them for the vertical one. To do so, we consider test functions in the basis of $\mathbb{P}_2[z]$

$$\phi_\alpha(z) \in \{\phi_{1,\alpha}(z), \phi_{2,\alpha}(z), \phi_{3,\alpha}(z)\} = \left\{1, z - z_\alpha, \frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right\}, \tag{27}$$

and we perform the layer-averaging procedure of the momentum equations multiplied by the test functions (as much as needed). Note that it is an orthogonal basis in L^2-norm, which makes easier the algebra. Second, concerning the pressure variables, we use restrictions (12) together with the integrated incompressibility condition (see [15] for details), yielding the set of 3L constraints

$$\begin{cases} \varphi_\alpha = -\partial_x\overline{u}_\alpha + \lambda_\alpha\partial_x z_\alpha, & \alpha = 1,\dots,L. \\ \psi_\alpha = -\partial_x\lambda_\alpha, & \alpha = 1,\dots,L, \\ \overline{w}_{\alpha+1} - \frac{h_{\alpha+1}\varphi_{\alpha+1}}{2} + \frac{h_{\alpha+1}^2\psi_{\alpha+1}}{12} - \overline{w}_\alpha - \frac{h_\alpha\varphi_\alpha}{2} - \frac{h_\alpha^2\psi_\alpha}{12} & \\ \qquad = \left(\overline{u}_{\alpha+1} - \frac{h_{\alpha+1}\lambda_{\alpha+1}}{2} - \overline{u}_\alpha - \frac{h_\alpha\lambda_\alpha}{2}\right)\partial_x z_{\alpha+1/2}, & \alpha = 1,\dots,L-1, \\ \partial_x\overline{u}_1 - \lambda_1\partial_x z_1 - \frac{h_1}{6}\partial_x\lambda_1 + \frac{\overline{w}_1 - w_{1/2}^+}{h_1/2} = 0 & \end{cases} \tag{28}$$

where $w^+_{1/2}$ is defined by the non-penetration condition

$$w^+_{1/2} = \left(\overline{u}_1 - \frac{h_1\lambda_1}{2}\right)\partial_x b.$$

We focus now on the evolution equations for the velocity variables. As commented above, the momentum conservation laws are multiplied by the test functions (27) and later integrated within each layer Ω_α. We remark that this is the same procedure developed in [15], and the terms that differs from the models introduced there are those coming from the stress tensor. Therefore, we focus here on these new contributions. Let us start by the horizontal momentum equation, where we write

$$\begin{aligned}
&\partial_t\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha\phi_{i,\alpha}\,\mathrm{d}z\right) + \partial_x\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha^2\phi_{i,\alpha}\,\mathrm{d}z\right) + \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \phi_{i,\alpha}\partial_x p_\alpha\,\mathrm{d}z - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha\partial_t\phi_{i,\alpha}\,\mathrm{d}z\\
&\quad - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha^2\partial_x\phi_{i,\alpha}\,\mathrm{d}z - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha w_\alpha\partial_z\phi_{i,\alpha}\,\mathrm{d}z = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \phi_{i,\alpha} g_x\,\mathrm{d}z + \partial_x\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\tau_{xx,\alpha}}{\rho}\phi_{i,\alpha}\,\mathrm{d}z\right)\\
&\quad - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\tau_{xx,\alpha}}{\rho}\partial_x\phi_{i,\alpha}\,\mathrm{d}z - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\tau_{xz,\alpha}}{\rho}\partial_z\phi_{i,\alpha}\,\mathrm{d}z - \phi_{i,\alpha|_{z_{\alpha+1/2}}}\left(K^-_{\alpha+1/2} + u^-_{\alpha+1/2}\Gamma_{\alpha+1/2}\right)\\
&\quad + \phi_{i,\alpha|_{z_{\alpha-1/2}}}\left(K^+_{\alpha-1/2} + u^+_{\alpha-1/2}\Gamma_{\alpha-1/2}\right),
\end{aligned}\tag{29}$$

with $i = 1, 2$. Notice that, using (17), the terms at the interfaces $\mathcal{L}_{\alpha\pm1/2}$ in (29) are written as

$$\phi_{i,\alpha|_{z_{\alpha-1/2}}}\big(K_{\alpha-1/2} + \widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2}\big) - \phi_{i,\alpha|_{z_{\alpha+1/2}}}\big(K_{\alpha+1/2} + \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2}\big),$$

with

$$\widetilde{u}_{\alpha+1/2} = \frac{u^-_{\alpha+1/2} + u^+_{\alpha+1/2}}{2} = \frac{\overline{u}_\alpha + \overline{u}_{\alpha+1}}{2} - \frac{h_{\alpha+1}\lambda_{\alpha+1} - h_\alpha\lambda_\alpha}{4}.$$

Thus, from (29), for $i = 1$ we have

$$\frac{1}{\rho}\left[\partial_x\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{xx,\alpha}\phi_{1,\alpha}\,\mathrm{d}z\right) - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{xx,\alpha}\partial_x\phi_{1,\alpha}\,\mathrm{d}z - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{xz,\alpha}\partial_z\phi_{1,\alpha}\,\mathrm{d}z\right] = \partial_x\left(h_\alpha\frac{\overline{\tau}_{xx,\alpha}}{\rho}\right),$$

since $\partial_x\phi_{1,\alpha} = \partial_z\phi_{1,\alpha} = 0$. For the conservation law for $h_\alpha\lambda_\alpha$, taking $\phi_{i,\alpha} = \phi_{2,\alpha}$ in (29), the integrals involving the stress tensor give

$$\begin{aligned}
&\frac{1}{\rho}\left[\frac{1}{h_\alpha}\partial_x\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{xx,\alpha}(z - z_\alpha)\,\mathrm{d}z\right) + \partial_x z_\alpha\overline{\tau}_{xx,\alpha} - \overline{\tau}_{xz,\alpha},\right]\\
&\qquad = \frac{1}{\rho}\left[\partial_x\left(\frac{h_\alpha^2\zeta_{xx,\alpha}}{12}\right) + \frac{h_\alpha\zeta_{xx,\alpha}}{12}\partial_x h_\alpha + \partial_x z_\alpha\overline{\tau}_{xx,\alpha} - \overline{\tau}_{xz,\alpha}\right].
\end{aligned}$$

Concerning the variables related to the vertical velocity, we obtain the equations for the evolution of $h_\alpha\overline{w}_\alpha, h_\alpha^2\varphi_\alpha, h_\alpha^3\psi_\alpha$ from the vertical momentum equation multiplied by the test functions in (27). In general, we consider the equation

$$\begin{aligned}
&\partial_t\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} w_\alpha\phi_{i,\alpha}\,\mathrm{d}z\right) + \partial_x\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha w_\alpha\phi_{i,\alpha}\,\mathrm{d}z\right) + \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \phi_{i,\alpha}\partial_z q_\alpha\,\mathrm{d}z - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} w_\alpha\partial_t\phi_{i,\alpha}\,\mathrm{d}z\\
&\quad - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha w_\alpha\partial_x\phi_{i,\alpha}\,\mathrm{d}z - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} w_\alpha^2\partial_z\phi_{i,\alpha}\,\mathrm{d}z = \partial_x\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\tau_{xz,\alpha}}{\rho}\phi_{i,\alpha}\,\mathrm{d}z\right) - \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\tau_{xz,\alpha}}{\rho}\partial_x\phi_{i,\alpha}\,\mathrm{d}z
\end{aligned}$$

$$- \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\tau_{zz,\alpha}}{\rho} \partial_z \phi_{i,\alpha} \, \mathrm{d}z - \phi_{i,\alpha|_{z_{\alpha+1/2}}} \left(K^-_{w,\alpha+1/2} + w^-_{\alpha+1/2} \Gamma_{\alpha+1/2} \right) + \phi_{i,\alpha|_{z_{\alpha-1/2}}} \left(K^+_{w,\alpha-1/2} + w^+_{\alpha-1/2} \Gamma_{\alpha-1/2} \right),$$

with $i = 1, 2, 3$. Note that in previous equation the hydrostatic contribution has been cancelled with the source term g_z, and analogously to the horizontal case, the terms at the interfaces are rewritten as

$$\phi_{i,\alpha|_{z_{\alpha-1/2}}} \left(K_{w,\alpha-1/2} + \widetilde{w}_{\alpha-1/2} \Gamma_{\alpha-1/2} \right) - \phi_{i,\alpha|_{z_{\alpha+1/2}}} \left(K_{w,\alpha+1/2} + \widetilde{w}_{\alpha+1/2} \Gamma_{\alpha+1/2} \right),$$

with

$$\widetilde{w}_{\alpha+1/2} = \frac{w^-_{\alpha+1/2} + w^+_{\alpha+1/2}}{2} = \frac{\overline{w}_\alpha + \overline{w}_{\alpha+1}}{2} - \frac{h_{\alpha+1}\varphi_{\alpha+1} - h_\alpha \varphi_\alpha}{4} + \frac{h_\alpha^2 \psi_\alpha + h_{\alpha+1}^2 \psi_{\alpha+1}}{24}.$$

Following the same layer-averaging procedure as previously for each test function $\phi_{i,\alpha}$, after some straightforward computations we get the evolution equation for $h_\alpha \overline{w}_\alpha$, $h_\alpha^2 \varphi_\alpha$ and $h_\alpha^3 \psi_\alpha$ for $\alpha = 1, \ldots, L$ (see (30)).

Hence, the final non-hydrostatic layer-averaged Navier–Stokes system LIN-NH$_2$-STRESS comprises 8L+1 equations together with restrictions (28). Namely, it reads

$$\partial_t H + \partial_x (H \bar{\bar{u}}) = 0, \tag{30a}$$

$$\begin{aligned} &\partial_t (h_\alpha \overline{u}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + \frac{h_\alpha^3 \lambda_\alpha^2}{12} \right) + |g_z| h_\alpha \partial_x (z_b + H) + \partial_x (h_\alpha \overline{q}_\alpha) = q_{\alpha+1/2} \partial_x z_{\alpha+1/2} \\ &\quad - q_{\alpha-1/2} \partial_x z_{\alpha-1/2} + \partial_x \left(h_\alpha \frac{\overline{\tau}_{xx,\alpha}}{\rho} \right) + K_{\alpha-1/2} - K_{\alpha+1/2} + \widetilde{u}_{\alpha-1/2} \Gamma_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2} \Gamma_{\alpha+1/2}, \end{aligned} \tag{30b}$$

$$\begin{aligned} &\partial_t \left(\frac{h_\alpha^2 \lambda_\alpha}{12} \right) + \partial_x \left(\frac{h_\alpha^2 \lambda_\alpha \overline{u}_\alpha}{12} + \frac{h_\alpha \left(q_{\alpha+1/2} - q_{\alpha-1/2} \right)}{20} + \frac{h_\alpha \pi_\alpha}{30} \right) + \frac{h_\alpha^2 \lambda_\alpha}{12} \partial_x \overline{u}_\alpha + \overline{q}_\alpha \partial_x z_\alpha \\ &\quad = - \left(\frac{(q_{\alpha+1/2} - q_{\alpha-1/2})}{20} + \frac{\pi_\alpha}{30} \right) \partial_x h_\alpha + \frac{1}{2} \left(q_{\alpha+1/2} \partial_x z_{\alpha+1/2} + q_{\alpha-1/2} \partial_x z_{\alpha-1/2} \right) \\ &\qquad + \partial_x \left(\frac{h_\alpha^2 \zeta_{xx,\alpha}}{12\rho} \right) + \frac{h_\alpha \zeta_{xx,\alpha}}{12\rho} \partial_x h_\alpha + \frac{1}{\rho} (\overline{\tau}_{xx,\alpha} \partial_x z_\alpha - \overline{\tau}_{xz,\alpha}) - \frac{1}{2} \left(K_{\alpha+1/2} + K_{\alpha-1/2} \right) \\ &\qquad - \Gamma_{\alpha-1/2} \left(\frac{h_\alpha \lambda_\alpha}{12} - \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha-1/2}}{2} \right) + \Gamma_{\alpha+1/2} \left(\frac{h_\alpha \lambda_\alpha}{12} + \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha+1/2}}{2} \right), \end{aligned} \tag{30c}$$

$$\begin{aligned} &\partial_t (h_\alpha \overline{w}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha + \frac{h_\alpha^3 \varphi_\alpha \lambda_\alpha}{12} \right) = q_{\alpha-1/2} - q_{\alpha+1/2} \\ &\quad + \partial_x \left(h_\alpha \frac{\overline{\tau}_{xz,\alpha}}{\rho} \right) + K_{w,\alpha-1/2} - K_{w,\alpha+1/2} + \widetilde{w}_{\alpha-1/2} \Gamma_{\alpha-1/2} - \widetilde{w}_{\alpha+1/2} \Gamma_{\alpha+1/2}, \end{aligned} \tag{30d}$$

$$\begin{aligned} &\partial_t \left(\frac{h_\alpha^2 \varphi_\alpha}{12} \right) + \partial_x \left(\frac{h_\alpha^2 \varphi_\alpha \overline{u}_\alpha}{12} + \frac{h_\alpha^4 \lambda_\alpha \psi_\alpha}{360} \right) + \frac{h_\alpha^2 \lambda_\alpha}{12} \partial_x \overline{w}_\alpha - \frac{h_\alpha^4 \psi_\alpha^2}{720} + \frac{h_\alpha^3 \lambda_\alpha \psi_\alpha}{360} \partial_x h_\alpha \\ &\quad + \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} - \overline{q}_\alpha = \partial_x \left(\frac{h_\alpha^2 \zeta_{xz,\alpha}}{12\rho} \right) + \frac{h_\alpha \zeta_{xz,\alpha}}{12\rho} \partial_x h_\alpha + \frac{1}{\rho} (\overline{\tau}_{xz,\alpha} \partial_x z_\alpha - \overline{\tau}_{zz,\alpha}) \\ &\quad - \frac{1}{2} \left(K_{w,\alpha-1/2} + K_{w,\alpha+1/2} \right) - \Gamma_{\alpha-1/2} \left(\frac{h_\alpha \varphi_\alpha}{12} - \frac{\overline{w}_\alpha - \widetilde{w}_{\alpha-1/2}}{2} \right) \\ &\quad + \Gamma_{\alpha+1/2} \left(\frac{h_\alpha \varphi_\alpha}{12} + \frac{\overline{w}_\alpha - \widetilde{w}_{\alpha+1/2}}{2} \right), \end{aligned} \tag{30e}$$

$$\begin{aligned} &\partial_t \left(\frac{h_\alpha^3 \psi_\alpha}{720} \right) + \partial_x \left(\frac{h_\alpha^3 \psi_\alpha \overline{u}_\alpha}{720} + \frac{h_\alpha^3 \varphi_\alpha \lambda_\alpha}{360} \right) - \frac{h_\alpha^2 \lambda_\alpha \varphi_\alpha}{120} \partial_x h_\alpha + \frac{h_\alpha^3 \psi_\alpha \varphi_\alpha}{240} = - \frac{\left(q_{\alpha+1/2} - q_{\alpha-1/2} \right) - \pi_\alpha}{30} \\ &\quad + \partial_x \left(\frac{h_\alpha^3 \xi_{xz,\alpha}}{720\rho} \right) + \frac{h_\alpha^2 \xi_{xz,\alpha}}{360\rho} \partial_x h_\alpha + \frac{h_\alpha}{12\rho} (\zeta_{xz,\alpha} \partial_x z_\alpha - \zeta_{zz,\alpha}) + \frac{\overline{\tau}_{xz,\alpha}}{12\rho} \partial_x h_\alpha \end{aligned}$$

$$
-\frac{1}{12}\big(K_{w,\alpha+1/2}-K_{w,\alpha-1/2}\big)-\Gamma_{\alpha-1/2}\bigg(\frac{h_\alpha^2\psi_\alpha}{360}-\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{12}\bigg)
+\Gamma_{\alpha+1/2}\bigg(\frac{h_\alpha^2\psi_\alpha}{360}+\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{12}\bigg), \tag{30f}
$$

for $\alpha=1,\dots,L$, combined with the following constraints,

$$
\begin{cases}
\varphi_\alpha=-\partial_x\overline{u}_\alpha+\lambda_\alpha\partial_x z_\alpha, & \alpha=1,\dots,L.\\
\psi_\alpha=-\partial_x\lambda_\alpha, & \alpha=1,\dots,L,\\
\overline{w}_{\alpha+1}-\frac{h_{\alpha+1}\varphi_{\alpha+1}}{2}+\frac{h_{\alpha+1}^2\psi_{\alpha+1}}{12}-\overline{w}_\alpha-\frac{h_\alpha\varphi_\alpha}{2}-\frac{h_\alpha^2\psi_\alpha}{12} & \\
\qquad=\Big(\overline{u}_{\alpha+1}-\frac{h_{\alpha+1}\lambda_{\alpha+1}}{2}-\overline{u}_\alpha-\frac{h_\alpha\lambda_\alpha}{2}\Big)\partial_x z_{\alpha+1/2}, & \alpha=1,\dots,L-1,\\
\partial_x\overline{u}_1-\lambda_1\partial_x z_1-\frac{h_1}{6}\partial_x\lambda_1+\frac{\overline{w}_1-w_{1/2}^+}{h_1/2}=0. &
\end{cases} \tag{31}
$$

This model satisfies the following energy balance.

Theorem 1. *Let us consider the LIN-NH$_2$-STRESS model defined by (30) and (31), with the stress tensor components defined by (13), and the terms $K_{\alpha\pm1/2},K_{w,\alpha\pm1/2}$ given by (18). The following energy balance is satisfied*

$$
\partial_t\bigg(\sum_{\alpha=1}^N E_\alpha\bigg)+\partial_x\Bigg[\sum_{\alpha=1}^N\bigg(\overline{u}_\alpha\bigg(E_\alpha+|g_z|h_\alpha\frac{h}{2}+\frac{h_\alpha^3\lambda_\alpha^2}{12}+h_\alpha\overline{q}_\alpha\bigg)+\frac{h_\alpha^3\lambda_\alpha\varphi_\alpha\overline{w}_\alpha}{12}+\frac{h_\alpha^5\lambda_\alpha\varphi_\alpha\psi_\alpha}{360}+\lambda_\alpha\frac{h_\alpha^2\pi_\alpha}{30}
$$
$$
+\lambda_\alpha\frac{h_\alpha^2(q_{\alpha+1/2}-q_{\alpha-1/2})}{20}-\frac{1}{\rho}\bigg(h_\alpha\overline{u}_\alpha\overline{\tau}_{xx,\alpha}+\frac{h_\alpha^3\lambda_\alpha\zeta_{xx,\alpha}}{12}+h_\alpha\overline{w}_\alpha\overline{\tau}_{xz,\alpha}+\frac{h_\alpha^3\varphi_\alpha\zeta_{xz,\alpha}}{12}+\frac{h_\alpha^5\psi_\alpha\xi_{xz,\alpha}}{720}\bigg)\bigg)\Bigg]
$$
$$
\le-\frac{1}{\rho}\sum_{\alpha=1}^N\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\bigg[\tau_{xx,\alpha}(z)\Big(\overline{[\partial_x u]}+\partial_x\lambda_\alpha(z-z_\alpha)\Big)
$$
$$
+\tau_{xz,\alpha}(z)\bigg(\overline{[\partial_z u]}_\alpha+\overline{[\partial_x w]}+(\partial_x\varphi_\alpha-\psi_\alpha\partial_x z_\alpha)(z-z_\alpha)+\partial_x\psi_\alpha\bigg(\frac{(z-z_\alpha)^2}{2}-\frac{h_\alpha^2}{24}\bigg)\bigg)
$$
$$
+\tau_{zz,\alpha}(z)\Big(\overline{[\partial_z w]}_\alpha+\psi_\alpha(z-z_\alpha)\Big)\bigg]\,\mathrm{d}z-\frac{1}{\rho}\bigg(\beta_0+\frac{\beta_1}{|\boldsymbol{U}|}\bigg)\Big(1+(\partial_x b)^2\Big)^{3/2}\Big(u_{1/2}^+\Big)^2 \tag{32}
$$

where

$$
E_\alpha\ :=\ h_\alpha\bigg(\frac{\overline{u}_\alpha^2+\overline{w}_\alpha^2}{2}+\frac{(h_\alpha\lambda_\alpha)^2+(h_\alpha\varphi_\alpha)^2}{24}+\frac{\big(h_\alpha^2\psi_\alpha\big)^2}{1440}+|g_z|\bigg(z_b+\frac{h}{2}\bigg)\bigg).
$$

Note that in previous result we do not claim that the right-hand side is non-positive. Actually, it is a general energy balance, and we need to consider a specific rheology defining the stress tensor in order to prove that it is indeed dissipative. It means that any definition of the stress tensor that leads to a non-positive right-hand side in (32) would be appropriate, from the point of view of the energy conservation. Before proving this theorem, let us introduce the result that is achieved in the particular case developed in Section 3.2, where the stress tensor is proportional to the strain rate tensor.

Corollary 1. *Let us consider a stress tensor proportional to the strain rate tensor, defined by (21), (23) and (25). Then, LIN-NH$_2$-STRESS model satisfies the dissipative energy balance (32), where the right-hand side is non-positive, being*

$$
-\frac{1}{\rho}\sum_{\alpha=1}^N\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\bigg[\big(\nu_{xx,\alpha}^0+\nu_{xx,\alpha}^1(z-z_\alpha)\big)\Big(\overline{[\partial_x u]}+\partial_x\lambda_\alpha(z-z_\alpha)\Big)^2
$$

$$+\frac{\left(\nu^0_{xz,\alpha}+\nu^1_{xz,\alpha}(z-z_\alpha)\right)}{2}\left(\overline{[\partial_z u]}_\alpha+\overline{[\partial_x w]}+(\partial_x\varphi_\alpha-\psi_\alpha\partial_x z_\alpha)(z-z_\alpha)+\partial_x\psi_\alpha\left(\frac{(z-z_\alpha)^2}{2}-\frac{h_\alpha^2}{24}\right)\right)^2$$

$$+\left(\nu^0_{zz,\alpha}+\nu^1_{zz,\alpha}(z-z_\alpha)\right)\left(\overline{[\partial_z w]}_\alpha+\psi_\alpha(z-z_\alpha)\right)^2\Bigg]\,\mathrm{d}z-\frac{1}{\rho}\left(\beta_0+\frac{\beta_1}{|\boldsymbol{U}|}\right)\left(1+(\partial_x b)^2\right)^{3/2}\left(u^+_{1/2}\right)^2\le 0.$$

Proof of Corollary 1. The proof follows from Theorem 1 and the definitions of the stress tensor components (21), (23) and (25). □

Proof of Theorem 1. We remark that model (30) and (31) differs from the LIN-NH$_2$ model introduced in [15], for which a exact energy balance was proven, only in the terms related to the deviatoric stress tensor. Therefore, we detail here how the inclusion of viscous terms leads to the energy balance above.

Defining

$$E_\alpha=h_\alpha\left(\frac{\overline{u}_\alpha^2+\overline{w}_\alpha^2}{2}+\frac{(h_\alpha\lambda_\alpha)^2+(h_\alpha\varphi_\alpha)^2}{24}+\frac{\left(h_\alpha^2\psi_\alpha\right)^2}{1440}+|g_z|\left(z_b+\frac{h}{2}\right)\right),$$

the energy balance is written

$$\partial_t E_\alpha+\partial_x\left(E_\alpha\overline{u}_\alpha+gh_\alpha\frac{h}{2}\overline{u}_\alpha+\frac{h_\alpha^3\lambda_\alpha^2\overline{u}_\alpha}{12}+\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha\overline{w}_\alpha}{12}+\frac{h_\alpha^5\varphi_\alpha\lambda_\alpha\psi_\alpha}{360}\right)+P_{\mathrm{NH},\alpha}$$
$$=V_\alpha+\mathrm{MT}_\alpha+\frac{|g_z|}{2}(h\partial_t h_\alpha-h_\alpha\partial_t h),$$

where

$$P_{\mathrm{NH},\alpha}=\partial_x\left(h_\alpha\overline{q}_\alpha\overline{u}_\alpha+\lambda_\alpha\left(\frac{h_\alpha^2\left(q_{\alpha+1/2}-q_{\alpha-1/2}\right)}{20}+\frac{h_\alpha^2\pi_\alpha}{30}\right)\right)$$
$$+q_{\alpha+1/2}\left(\partial_t z_{\alpha+1/2}-\Gamma_{\alpha+1/2}\right)-q_{\alpha-1/2}\left(\partial_t z_{\alpha-1/2}-\Gamma_{\alpha-1/2}\right)$$

are the non-hydrostatic contributions, satisfying that

$$\sum_{\beta=1}^N P_{\mathrm{NH},\beta}=\sum_{\beta=1}^N\partial_x\left(h_\beta\overline{q}_\beta\overline{u}_\beta+\lambda_\beta\left(\frac{h_\beta^2(\delta q)_\beta}{20}+\frac{h_\beta^2\pi_\beta}{30}\right)\right),$$

when summing up to α. The terms involving mass transference MT_α are

$$\mathrm{MT}_\alpha=-\frac{\Gamma_{\alpha+1/2}}{2}\left(u^-_{\alpha+1/2}u^+_{\alpha+1/2}+w^-_{\alpha+1/2}w^+_{\alpha+1/2}\right)+\frac{\Gamma_{\alpha-1/2}}{2}\left(u^-_{\alpha-1/2}u^+_{\alpha-1/2}+w^-_{\alpha-1/2}w^+_{\alpha-1/2}\right)$$
$$-|g_z|(z_b+h)\left(\Gamma_{\alpha+1/2}-\Gamma_{\alpha-1/2}\right),$$

where one can check that

$$\sum_{\beta=1}^N\mathrm{MT}_\beta=0,\qquad\text{and}\qquad\frac{|g_z|}{2}\sum_{\beta=1}^N(h\partial_t h_\alpha-h_\alpha\partial_t h)=0.$$

Finally, the viscous terms are

$$\rho V_\alpha=\overline{u}_\alpha\partial_x(h_\alpha\overline{\tau}_{xx,\alpha})+\lambda_\alpha\partial_x\left(\frac{h_\alpha^3\zeta_{xx,\alpha}}{12}\right)+h_\alpha\lambda_\alpha(\overline{\tau}_{xx,\alpha}\partial_x z_\alpha-\overline{\tau}_{xz,\alpha})+\overline{w}_\alpha\partial_x(h_\alpha\overline{\tau}_{xz,\alpha})+\varphi_\alpha\partial_x\left(\frac{h_\alpha^3\zeta_{xz,\alpha}}{12}\right)$$
$$+h_\alpha\varphi_\alpha(\overline{\tau}_{xz,\alpha}\partial_x z_\alpha-\overline{\tau}_{zz,\alpha})+\psi_\alpha\partial_x\left(\frac{h_\alpha^5\xi_{xz,\alpha}}{720}\right)+\frac{h_\alpha^3\psi_\alpha}{12}(\zeta_{xz,\alpha}\partial_x z_\alpha-\zeta_{zz,\alpha})+\frac{h_\alpha^2\psi_\alpha}{12}\overline{\tau}_{xz,\alpha}\partial_x h_\alpha$$

$$- \rho u^-_{\alpha+1/2} K_{\alpha+1/2} + \rho u^+_{\alpha-1/2} K_{\alpha-1/2} - \rho w^-_{\alpha+1/2} K_{w,\alpha+1/2} + \rho w^+_{\alpha-1/2} K_{w,\alpha-1/2},$$

where $K_{\alpha\pm1/2}, K_{w,\alpha\pm1/2}$ are defined by (18) and $u^\mp_{\alpha\pm1/2}, w^\mp_{\alpha\pm1/2}$ by (8) and (11), respectively. We focus now on the terms appearing at each layer Ω_α when summing up α. In particular, when adding $V_\alpha + V_{\alpha+1}$, it makes appear the term

$$[u]_{\alpha+1/2} K_{\alpha+1/2} + [w]_{\alpha+1/2} K_{w,\alpha+1/2}.$$

Notice now that the definitions (18a) makes these jumps at the interfaces to be equally distributed to the upper $(\Omega_{\alpha+1})$ and lower (Ω_α) layers.

Thus, the viscous term at each layer Ω_α is divided as

$$V_\alpha = V_{1,\alpha} + V_{2,\alpha},$$

where $V_{1,\alpha}$ is the conservative term

$$\rho\, V_{1,\alpha} = \partial_x \left(h_\alpha \overline{u}_\alpha \overline{\tau}_{xx,\alpha} + \frac{h_\alpha^3 \lambda_\alpha \zeta_{xx,\alpha}}{12} + h_\alpha \overline{w}_\alpha \overline{\tau}_{xz,\alpha} + \frac{h_\alpha^3 \varphi_\alpha \zeta_{xz,\alpha}}{12} + \frac{h_\alpha^5 \psi_\alpha \xi_{xz,\alpha}}{720} \right),$$

and $V_{2,\alpha}$ collects the rest of terms, including those at the interfaces. This second term takes the form

$$\begin{aligned}
\rho\, V_{2,\alpha} = & -\frac{h_\alpha^3}{12}(\zeta_{xx,\alpha}\partial_x\lambda_\alpha + \zeta_{zz,\alpha}\psi_\alpha) - \frac{h_\alpha^3 \zeta_{xz,\alpha}}{12}(\partial_x\varphi_\alpha - \psi_\alpha \partial_x z_\alpha) - \frac{h_\alpha^5 \xi_{xz,\alpha}}{720}\partial_x \psi_\alpha \\
& - \overline{\tau}_{xx,\alpha}\left(h_\alpha \partial_x \overline{u}_\alpha - h_\alpha \lambda_\alpha \partial_x z_\alpha - \frac{[u]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [u]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2} \right) \\
& - \overline{\tau}_{zz,\alpha}\left(h_\alpha \varphi_\alpha + \frac{[w]_{\alpha+1/2} + [w]_{\alpha-1/2}}{2} \right) - \overline{\tau}_{xz,\alpha}\left(h_\alpha \lambda_\alpha + \frac{[u]_{\alpha+1/2} + [u]_{\alpha-1/2}}{2} \right) \\
& - \overline{\tau}_{xz,\alpha}\left(h_\alpha\left(\partial_x \overline{w}_\alpha - \frac{h_\alpha \psi_\alpha}{12}\partial_x h_\alpha \right) - h_\alpha \varphi_\alpha \partial_x z_\alpha - \frac{[w]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [w]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2} \right),
\end{aligned}$$

where we immediately recognize the definitions given for the derivatives $\overline{[\partial_x u]}_\alpha$, $\overline{[\partial_x w]}_\alpha$, $\overline{[\partial_z u]}_\alpha$ and $\overline{[\partial_z w]}_\alpha$ (see (20)). It is now convenient to split V_α as the sum

$$V_{2,\alpha} = V_{2,xx,\alpha} + V_{2,xz,\alpha} + V_{2,zz,\alpha},$$

where each term on the right-hand side accounts for a component of the stress tensor. We can prove now, taking into account the definition of the stress tensor components, that each of the previous addends gives us one of the right-hand side terms in (32). Let us detail each of them. First, considering (22), we see that

$$\begin{aligned}
\rho V_{2,xx,\alpha} &= -\frac{h_\alpha^3}{12}\zeta_{xx,\alpha}\partial_x\lambda_\alpha - h_\alpha \overline{\tau}_{xx,\alpha}\overline{[\partial_x u]}_\alpha \\
&= -\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\overline{\tau}_{xx,\alpha} + \zeta_{xx,\alpha}(z - z_\alpha) + \xi_{xx,\alpha}\left(\frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24} \right) \right) \left(\overline{[\partial_x u]} + \partial_x \lambda_\alpha (z - z_\alpha) \right) \mathrm{d}z \\
&= -\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{xx,\alpha}(z) \left(\overline{[\partial_x u]} + \partial_x \lambda_\alpha (z - z_\alpha) \right) \mathrm{d}z.
\end{aligned}$$

Second, with (26) we reach

$$\begin{aligned}
\rho V_{2,zz,\alpha} &= -\frac{h_\alpha^3}{12}\zeta_{zz,\alpha}\psi_\alpha - h_\alpha \overline{\tau}_{zz,\alpha}\overline{[\partial_z w]}_\alpha \\
&= -\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\overline{\tau}_{zz,\alpha} + \zeta_{zz,\alpha}(z - z_\alpha) + \xi_{zz,\alpha}\left(\frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24} \right) \right) \left(\overline{[\partial_z w]}_\alpha + \psi_\alpha (z - z_\alpha) \right) \mathrm{d}z
\end{aligned}$$

$$= -\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{zz,\alpha}(z)\Big(\overline{[\partial_z w]}_\alpha + \psi_\alpha(z-z_\alpha)\Big)\,\mathrm{d}z.$$

Third, we obtain, thanks to (24),

$$\begin{aligned}\rho V_{2,xz,\alpha} &= -\frac{h_\alpha^3\zeta_{xz,\alpha}}{12}(\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha) - \frac{h_\alpha^5\xi_{xz,\alpha}}{720}\partial_x\psi_\alpha - \overline{\tau}_{xz,\alpha}h_\alpha\Big(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]}_\alpha\Big)\\ &= -\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\Bigg(\overline{\tau}_{xz,\alpha} + \zeta_{xz,\alpha}(z-z_\alpha) + \xi_{xz,\alpha}\bigg(\frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\bigg) + \varkappa_{ij,\alpha}\bigg(\frac{(z-z_\alpha)^3}{3} - \frac{h_\alpha^2}{20}(z-z_\alpha)\bigg)\Bigg)\\ &\quad\times\Bigg(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]} + (\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha)(z-z_\alpha) + \partial_x\psi_\alpha\bigg(\frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\bigg)\Bigg)\,\mathrm{d}z\\ &= -\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\tau_{xz,\alpha}(z)\Bigg(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]} + (\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha)(z-z_\alpha) + \partial_x\psi_\alpha\bigg(\frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\bigg)\Bigg)\,\mathrm{d}z.\end{aligned}$$

A special attention deserves the first layer Ω_1, where we get the term

$$u_{1/2}^{+}K_{1/2} + w_{1/2}^{+}K_{w,1/2},$$

which using the non-penetration condition and the friction condition (18c), is written as the dissipative term

$$u_{1/2}^{+}\big(K_{1/2} + K_{w,1/2}\partial_x b\big) = -\frac{1}{\rho}\bigg(\beta_0 + \frac{\beta_1}{|\boldsymbol{U}|}\bigg)\Big(1 + (\partial_x b)^2\Big)^{3/2}\Big(u_{1/2}^{+}\Big)^2,$$

what concludes the proof. □

4.2. Second-order approximation of the stress tensor

We remind the reader that the main goal of this work is to extend the models presented in [15] to the Navier–Stokes system. An important remark is the fact that, whereas the proposed models for Euler equations in the previous work are second-order accurate in the vertical direction, a layerwise linear horizontal velocity is not enough to ensure the second-order accuracy in the Navier–Stokes case.

Concretely, the vertical derivative of the horizontal velocity (λ_α) is layerwise constant. Then, we only get a first-order approximation of $\tau_{xz} = \nu_{xz}(\partial_z u + \partial_x w)/2$. In the next, we explain some strategies to overcome this problem.

A first option is to develop a more complicated model considering a parabolic vertical profile for the horizontal velocity. That would leads to a more complex model. An interesting second option is to keep the linear profile of u_α but improving only the approximation of $\tau_{xz,\alpha}$ by using an external approximation. Let us develop this second alternative in what follows.

Let us remark that the approximation in Ω_α of the rest of components of $\boldsymbol{\tau}$ do not need to be modified since they already are second-order approximations of each component ($\partial_x u_\alpha$ and $\partial_z w_\alpha$ are linear, and $\partial_x w_\alpha$ is a parabolic function). The problem is the fact that here $\partial_z u_\alpha$ is constant. Then, in order to get a second-order approximation of $\nu_{xz,\alpha}\partial_z u_\alpha$, we consider its Taylor expansion on $z = z_\alpha$:

$$(\nu_{xz,\alpha}\partial_z u_\alpha)(z) = (\nu_{xz,\alpha}\partial_z u_\alpha)_{|z=z_\alpha} + (\partial_z\nu_{xz,\alpha}\partial_z u_\alpha + \nu_{xz,\alpha}\partial_{zz}u_\alpha)_{|z=z_\alpha}(z-z_\alpha) + \mathcal{O}\big(h_\alpha^2\big).$$

Then, looking at (23) and taking into account that $\nu_{xz,\alpha}^1 = \partial_z\nu_{xz,\alpha}(z)_{|z=z_\alpha}$, in order to get the second-order approximation of $\tau_{xz,\alpha}$ we need to include the term $(\nu_{xz,\alpha}\partial_{zz}u_\alpha)_{|z=z_\alpha}$ in the definition of $\zeta_{xz,\alpha}$. Thus, in this case, the corrected stress tensor component, that we will denote by $\widetilde{\tau}_{xz,\alpha}$ hereinafter, is defined by (13) with (24), where the term $\zeta_{xz,\alpha}$ is replaced by

$$\widetilde{\zeta}_{xz,\alpha} = \tfrac{\nu_{xz,\alpha}^1}{2}\bigg(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]}_\alpha + \tfrac{h_\alpha^2}{30}\partial_x\psi_\alpha\bigg) + \tfrac{\nu_{xz,\alpha}^0}{2}(\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha + \widetilde{\chi}_\alpha), \tag{33a}$$

being $\widetilde{\chi}_\alpha$ an external approximation of the second-order derivative of u_α. For instance, we can choose

$$\widetilde{\chi}_\alpha = \frac{\overline{[\partial_z u]}_{\alpha+1} - \overline{[\partial_z u]}_{\alpha-1}}{h_\alpha + \frac{1}{2}(h_{\alpha+1} + h_{\alpha-1})} \qquad \text{for } \alpha = 2, \ldots, N-1, \tag{33b}$$

where one should pay attention to the case $\alpha = 1, N$. By simplicity, we consider here a second-order upwind finite difference approximation. Note that model (30) and (31) does not change, just the definition of $\zeta_{xz,\alpha}$ must be modified.

In this case, the model does not satisfy an exact dissipative balance as in Corollary 1. However, one can prove that the term that is not controlled is proportional to h_α^3. Concretely, this term written on the right-hand side is

$$\frac{-h_\alpha^3 \nu_{xz,\alpha}^0}{24} \widetilde{\chi}_\alpha (\partial_x \varphi_\alpha - \psi_\alpha \partial_x z_\alpha).$$

Taking into account that E_α is proportional to h_α, the corrected model, that uses (33), satisfies a dissipative energy balance up to second order.

In Section 6 the key role of this correction will be shown for some well-known geophysical flows in simple configurations.

4.3. Simplified model: LIN-NH$_1$-STRESS

As showed in [15], simplified models can be deduced by simplifying the vertical profile of the vertical velocity in the momentum conservation law. Remark that, in those models, this vertical profile is not simplified in the incompressibility equation, as explained in that previous work. There, the cases of linear and constant vertical velocity were presented and analysed. Two simplified non-hydrostatic models were presented, both satisfying a dissipative energy balance. However, only the one with layerwise linear vertical velocity (actually $u_\alpha \in \mathbb{P}_1$, $w_\alpha \in \mathbb{P}_1$, $q_\alpha \in \mathbb{P}_2$), denoted there LIN-NH$_1$ kept the good dispersive properties. Therefore, we present here the extension of this model to the Navier–Stokes case, denoted here as LIN-NH$_1$-STRESS.

For the sake of brevity we summarize here the main hypothesis of this model. The horizontal velocity is still linear (see (7)) while the vertical velocity and the non-hydrostatic pressure at the layer Ω_α are given in this case by

$$w_\alpha(z) = \overline{w}_\alpha + \varphi_\alpha(z - z_\alpha), \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}],$$

and

$$q_\alpha(z) = \frac{3\overline{q}_\alpha - \widehat{q}_\alpha}{2} + \pi_\alpha \frac{z - z_\alpha}{h_\alpha} + 6(\widehat{q}_\alpha - \overline{q}_\alpha)\frac{(z - z_\alpha)^2}{h_\alpha^2},$$

for $z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]$. Note that this simplification of the vertical velocity makes the stress tensor components to be also simplified. Let us consider the stress tensor in the form of Section 3.2. Namely, $\tau_{xz,\alpha}$ and $\tau_{zz,\alpha}$ are now parabolic and linear, respectively, whereas $\tau_{xx,\alpha}$ does not change its definition. Concretely, they are defined by (13) with (24), (26) and taking $\varkappa_{xz,\alpha} = \xi_{zz,\alpha} = 0$ and $\psi_\alpha = 0$. Let us detail them. Their profiles are

$$\begin{aligned} \tau_{xz,\alpha}(z) &= \overline{\tau}_{xz,\alpha} + \zeta_{xz,\alpha}(z - z_\alpha) + \xi_{xz,\alpha}\left(\frac{(z - z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right), \\ \tau_{zz,\alpha}(z) &= \overline{\tau}_{zz,\alpha} + \zeta_{zz,\alpha}(z - z_\alpha), \end{aligned} \tag{34a}$$

with

$$\begin{aligned} \overline{\tau}_{xz,\alpha} &= \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{xz,\alpha}(z)\,\mathrm{d}z = \frac{\nu_{xz,\alpha}^0}{2}\left(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]}_\alpha\right) + \frac{\nu_{xz,\alpha}^1 h_\alpha^2}{24}\partial_x \varphi_\alpha, \\ \zeta_{xz,\alpha} &= \frac{\nu_{xz,\alpha}^1}{2}\left(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]}_\alpha\right) + \frac{\nu_{xz,\alpha}^0}{2}\partial_x \varphi_\alpha, \end{aligned}$$

$$\xi_{xz,\alpha} = \nu^1_{xz,\alpha}\partial_x\varphi_\alpha, \tag{34b}$$

and

$$\begin{aligned}\overline{\tau}_{zz,\alpha} &= \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{zz,\alpha}(z)\,\mathrm{d}z = \nu^0_{zz,\alpha}\overline{[\partial_z w]}_\alpha,\\ \zeta_{zz,\alpha} &= \nu^1_{zz,\alpha}\overline{[\partial_z w]}_\alpha.\end{aligned} \tag{34c}$$

The final LIN-NH$_1$-STRESS model is obtained analogously to the previous model, taking into account these assumptions. However, it can also be deduced directly from (30) and (31) by making $\psi_\alpha = 0$, $\pi_\alpha = q_{\alpha+1/2} - q_{\alpha-1/2}$ and removing the last equation in (30), and accordingly rewriting the constraints. It comprises 6L+1 equations and unknowns ($\{H, \overline{u}_\alpha, \lambda_\alpha, \overline{w}_\alpha, \varphi_\alpha, \overline{q}_\alpha, q_{\alpha-1/2}\}$ for $\alpha = 1,\dots,L$), and reads

$$\partial_t H + \partial_x(H\bar{\overline{u}}) = 0, \tag{35a}$$

$$\begin{aligned}&\partial_t(h_\alpha\overline{u}_\alpha) + \partial_x\left(h_\alpha\overline{u}_\alpha^2 + \frac{h_\alpha^3\lambda_\alpha^2}{12}\right) + |g_z|h_\alpha\partial_x(z_b+H) + \partial_x(h_\alpha\overline{q}_\alpha) = q_{\alpha+1/2}\partial_x z_{\alpha+1/2} - q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\\ &\quad + \partial_x\left(h_\alpha\frac{\overline{\tau}_{xx,\alpha}}{\rho}\right) + K_{\alpha-1/2} - K_{\alpha+1/2} + \widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2}\end{aligned} \tag{35b}$$

$$\begin{aligned}&\partial_t\left(\frac{h_\alpha^2\lambda_\alpha}{12}\right) + \partial_x\left(\frac{h_\alpha^2\lambda_\alpha\overline{u}_\alpha}{12} + \frac{h_\alpha(q_{\alpha+1/2}-q_{\alpha-1/2})}{12}\right) + \frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{u}_\alpha + \overline{q}_\alpha\partial_x z_\alpha = -\frac{(q_{\alpha+1/2}-q_{\alpha-1/2})}{12}\partial_x h_\alpha\\ &\quad + \frac{1}{2}\big(q_{\alpha+1/2}\partial_x z_{\alpha+1/2} + q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\big) + \partial_x\left(\frac{h_\alpha^2\zeta_{xx,\alpha}}{12\rho}\right) + \frac{h_\alpha\zeta_{xx,\alpha}}{12\rho}\partial_x h_\alpha + \frac{1}{\rho}(\overline{\tau}_{xx,\alpha}\partial_x z_\alpha - \overline{\tau}_{xz,\alpha})\\ &\quad - \frac{1}{2}\big(K_{\alpha+1/2} + K_{\alpha-1/2}\big) - \Gamma_{\alpha-1/2}\left(\frac{h_\alpha\lambda_\alpha}{12} - \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha-1/2}}{2}\right) + \Gamma_{\alpha+1/2}\left(\frac{h_\alpha\lambda_\alpha}{12} + \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha+1/2}}{2}\right)\end{aligned} \tag{35c}$$

$$\begin{aligned}&\partial_t(h_\alpha\overline{w}_\alpha) + \partial_x\left(h_\alpha\overline{u}_\alpha\overline{w}_\alpha + \frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{12}\right)\\ &\quad = q_{\alpha-1/2} - q_{\alpha+1/2} + \partial_x\left(h_\alpha\frac{\overline{\tau}_{xz,\alpha}}{\rho}\right) + K_{w,\alpha-1/2} - K_{w,\alpha+1/2} + \widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2} - \widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2},\end{aligned} \tag{35d}$$

$$\begin{aligned}&\partial_t\left(\frac{h_\alpha^2\varphi_\alpha}{12}\right) + \partial_x\left(\frac{h_\alpha^2\varphi_\alpha\overline{u}_\alpha}{12}\right) + \frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{w}_\alpha + \frac{q_{\alpha+1/2}+q_{\alpha-1/2}}{2} - \overline{q}_\alpha = \partial_x\left(\frac{h_\alpha^2\zeta_{xz,\alpha}}{12\rho}\right) + \frac{h_\alpha\zeta_{xz,\alpha}}{12\rho}\partial_x h_\alpha\\ &\quad + \frac{1}{\rho}(\overline{\tau}_{xz,\alpha}\partial_x z_\alpha - \overline{\tau}_{zz,\alpha}) - \frac{1}{2}\big(K_{w,\alpha-1/2} + K_{w,\alpha+1/2}\big) - \Gamma_{\alpha-1/2}\left(\frac{h_\alpha\varphi_\alpha}{12} - \frac{\overline{w}_\alpha - \widetilde{w}_{\alpha-1/2}}{2}\right)\\ &\quad + \Gamma_{\alpha+1/2}\left(\frac{h_\alpha\varphi_\alpha}{12} + \frac{\overline{w}_\alpha - \widetilde{w}_{\alpha+1/2}}{2}\right),\end{aligned} \tag{35e}$$

for $\alpha = 1,\dots,L$, with the velocities at the interfaces

$$\widetilde{u}_{\alpha+1/2} = \frac{\overline{u}_\alpha + \overline{u}_{\alpha+1}}{2} - \frac{h_{\alpha+1}\lambda_{\alpha+1} - h_\alpha\lambda_\alpha}{4}, \qquad \widetilde{w}_{\alpha+1/2} = \frac{\overline{w}_\alpha + \overline{w}_{\alpha+1}}{2} - \frac{h_{\alpha+1}\varphi_{\alpha+1} - h_\alpha\varphi_\alpha}{4},$$

combined with the following constraints, where $\psi_\alpha = -\partial_x\lambda_\alpha$ has been replaced on the averaged incompressibility in (31),

$$\begin{cases}\varphi_\alpha = -\partial_x\overline{u}_\alpha + \lambda_\alpha\partial_x z_\alpha, & \alpha = 1,\dots,L,\\ \overline{w}_{\alpha+1} - \frac{h_{\alpha+1}\varphi_{\alpha+1}}{2} - \frac{h_{\alpha+1}^2}{12}\partial_x\lambda_{\alpha+1} - \overline{w}_\alpha - \frac{h_\alpha\varphi_\alpha}{2} + \frac{h_\alpha^2}{12}\partial_x\lambda_\alpha & \\ \qquad = \left(\overline{u}_{\alpha+1} - \frac{h_{\alpha+1}\lambda_{\alpha+1}}{2} - \overline{u}_\alpha - \frac{h_\alpha\lambda_\alpha}{2}\right)\partial_x z_{\alpha+1/2}, & \alpha = 1,\dots,L-1,\\ \partial_x\overline{u}_1 - \lambda_1\partial_x z_1 - \frac{h_1}{6}\partial_x\lambda_1 + \frac{\overline{w}_1 - w_{1/2}^+}{h_1/2} = 0. & \end{cases} \tag{36}$$

This model also satisfies a dissipative energy balance, whose proof is analogous to that of Theorem 1:

Theorem 2. *Let us consider the LIN-NH$_1$-STRESS model ((35), (36)), with the stress tensor components defined by (22) and (34), and the terms $K_{\alpha\pm1/2}, K_{w,\alpha\pm1/2}$ given by (18). The following dissipative energy balance is satisfied*

$$\partial_t\left(\sum_{\alpha=1}^N E_\alpha\right)+\partial_x\left[\sum_{\alpha=1}^N\left(\overline{u}_\alpha\left(E_\alpha+|g_z|h_\alpha\frac{h}{2}+\frac{h_\alpha^3\lambda_\alpha^2}{12}+h_\alpha\overline{q}_\alpha\right)+\frac{h_\alpha^3\lambda_\alpha\varphi_\alpha\overline{w}_\alpha}{12}+\frac{h_\alpha^2\lambda_\alpha\left(q_{\alpha+1/2}-q_{\alpha-1/2}\right)}{12}\right.\right.$$
$$\left.\left.-\frac{1}{\rho}\left(h_\alpha\overline{u}_\alpha\overline{\tau}_{xx,\alpha}+\frac{h_\alpha^3\lambda_\alpha\zeta_{xx,\alpha}}{12}+h_\alpha\overline{w}_\alpha\overline{\tau}_{xz,\alpha}+\frac{h_\alpha^3\varphi_\alpha\zeta_{xz,\alpha}}{12}\right)\right)\right]$$
$$\leq-\frac{1}{\rho}\sum_{\alpha=1}^N\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\left[\left(\nu_{xx,\alpha}^0+\nu_{xx,\alpha}^1(z-z_\alpha)\right)\left(\overline{[\partial_x u]}+\partial_x\lambda_\alpha(z-z_\alpha)\right)^2+\left(\nu_{zz,\alpha}^0+\nu_{zz,\alpha}^1(z-z_\alpha)\right)\left(\overline{[\partial_z w]}_\alpha\right)^2\right.$$
$$\left.+\frac{\left(\nu_{xz,\alpha}^0+\nu_{xz,\alpha}^1(z-z_\alpha)\right)}{2}\left(\overline{[\partial_z u]}_\alpha+\overline{[\partial_x w]}+\partial_x\varphi_\alpha(z-z_\alpha)\right)^2\right]\mathrm{d}z-\frac{1}{\rho}\left(\beta_0+\frac{\beta_1}{|\boldsymbol{U}|}\right)\left(1+(\partial_x b)^2\right)^{3/2}\left(u_{1/2}^+\right)^2$$

where

$$E_\alpha:=h_\alpha\left(\frac{\overline{u}_\alpha^2+\overline{w}_\alpha^2}{2}+\frac{(h_\alpha\lambda_\alpha)^2+(h_\alpha\varphi_\alpha)^2}{24}+|g_z|\left(z_b+\frac{h}{2}\right)\right).$$

4.4. Compact form of LIN-NH$_k$-STRESS models

The models presented in this work have to be analyzed before discretizing them. For instance, the ideas of [14], where a multilayer model with hydrostatic pressure, layerwise constant velocity and a simpler definition of the viscous terms is considered, may be followed to study the global stability of weak solutions. It is also useful to study the structure of the resulting PDE systems, especially in order to address their numerical approximation. In this subsection we rewrite the LIN-NH$_2$-STRESS model in a compact way. The LIN-NH$_1$-STRESS model is also written in this compact form as a particular case, as we explain at the end of this subsection.

Let us consider the notation introduced in [15],

$$\Lambda_\alpha=\frac{h_\alpha\lambda_\alpha}{2\sqrt{3}},\quad \Phi_\alpha=\frac{h_\alpha\varphi_\alpha}{2\sqrt{3}},\quad \Psi_\alpha=\frac{h_\alpha^2\psi_\alpha}{12\sqrt{5}},$$

and we set the following notation for the stress tensor components,

$$\mathcal{Z}_{xx,\alpha}=\frac{h_\alpha\zeta_{xx,\alpha}}{2\sqrt{3}},\quad \mathcal{Z}_{xz,\alpha}=\frac{h_\alpha\zeta_{xz,\alpha}}{2\sqrt{3}},\quad \mathcal{R}_{xz,\alpha}=\frac{h_\alpha\xi_{xz,\alpha}}{12\sqrt{5}}.$$

Then, system (30) and constraints (31) may be written in the following form:

$$\begin{cases}\partial_t H+\partial_x(H\bar{\bar{u}})=0,\\ \partial_t(h_\alpha\mathbf{X}_\alpha)+\partial_x(h_\alpha\mathbf{X}_\alpha\overline{u}_\alpha)+\mathbf{F}_\alpha+\nabla_{\mathrm{NH}}\mathbf{Q}_\alpha=\mathbf{S}_\alpha\partial_x(z_b+H)+\partial_x\mathbf{D}_{\tau,\alpha}\\ \qquad\qquad+\Gamma_{\alpha+1/2}\mathbf{G}_\alpha^+-\Gamma_{\alpha-1/2}\mathbf{G}_\alpha^-+\mathbf{G}_{\tau,\alpha}^+-\mathbf{G}_{\tau,\alpha}^-,\\ \nabla_{\mathrm{NH}}\cdot\mathbf{X}_\alpha=0,\end{cases}$$

where the vectors of unknowns are

$$\mathbf{X}_\alpha=(\overline{u}_\alpha,\Lambda_\alpha,\overline{w}_\alpha,\Phi_\alpha,\Psi_\alpha)',\qquad \mathbf{Q}_\alpha=\left(\overline{q}_\alpha,q_{\alpha-1/2},\pi_\alpha\right)',$$

the vectors $\mathbf{F}_\alpha$, $\mathbf{S}_\alpha$ and $\mathbf{G}_\alpha^\pm$ are defined as in [15] for LIN-NH$_2$ model, *i.e.* $\mathbf{S}_\alpha = (-gh_\alpha, 0, 0, 0, 0)'$,

$$\mathbf{F}_\alpha = \begin{pmatrix} \partial_x\big(h_\alpha \Lambda_\alpha^2\big) \\ h_\alpha \Lambda_\alpha \partial_x \overline{u}_\alpha \\ \partial_x(h_\alpha \Lambda_\alpha \Phi_\alpha) \\ \frac{2\sqrt{5}}{5}\partial_x(h_\alpha\Lambda_\alpha\Psi_\alpha) - 2\sqrt{3}\Psi_\alpha^2 + h_\alpha\Lambda_\alpha\partial_x\overline{w}_\alpha + \frac{2\sqrt{5}}{5}\Lambda_\alpha\Psi_\alpha\partial_x h_\alpha \\ \frac{2\sqrt{5}}{5}\partial_x(h_\alpha\Lambda_\alpha\Phi_\alpha) - \frac{6\sqrt{5}}{5}\Lambda_\alpha\Phi_\alpha\partial_x h_\alpha + 6\sqrt{3}\Phi_\alpha\Psi_\alpha \end{pmatrix},$$

$$\mathbf{G}_\alpha^\pm = \begin{pmatrix} -\widetilde{u}_{\alpha\pm 1/2} \\ \Lambda_\alpha \pm \sqrt{3}\big(\overline{u}_\alpha - \widetilde{u}_{\alpha\pm 1/2}\big) \\ -\widetilde{w}_{\alpha\pm 1/2} \\ \Phi_\alpha \pm \sqrt{3}\big(\overline{w}_\alpha - \widetilde{w}_{\alpha\pm 1/2}\big) \\ 2\Psi_\alpha \pm \sqrt{15}\Phi_\alpha + \sqrt{5}\big(\overline{w}_\alpha - \widetilde{w}_{\alpha\pm 1/2}\big) \end{pmatrix},$$

and the terms related to the stress tensor components are defined as follows:

$$\mathbf{D}_{\tau,\alpha} = \frac{1}{\rho}\begin{pmatrix} h_\alpha \overline{\tau}_{xx,\alpha} \\ h_\alpha \mathcal{Z}_{xx,\alpha} \\ h_\alpha \overline{\tau}_{xz,\alpha} \\ h_\alpha \mathcal{Z}_{xz,\alpha} \\ h_\alpha \mathcal{R}_{xz,\alpha} \end{pmatrix}, \quad \mathbf{G}_{\tau,\alpha}^\pm = \frac{1}{\rho}\begin{pmatrix} -\rho K_{\alpha\pm 1/2} \\ \mathcal{Z}_{xx,\alpha}\partial_x z_{\alpha\pm 1/2} \pm \sqrt{3}\big((\overline{\tau}_{xx,\alpha}\partial_x z_\alpha - \overline{\tau}_{xz,\alpha}) - \rho K_{\alpha\pm 1/2}\big) \\ -\rho K_{w,\alpha\pm 1/2} \\ \mathcal{Z}_{xz,\alpha}\partial_x z_{\alpha\pm 1/2} \pm \sqrt{3}\big((\overline{\tau}_{xz,\alpha}\partial_x z_\alpha - \overline{\tau}_{zz,\alpha}) - \rho K_{w,\alpha\pm 1/2}\big) \\ 2\mathcal{R}_{xz,\alpha} \pm \sqrt{15}(\mathcal{Z}_{xz,\alpha}\partial_x z_\alpha - \mathcal{Z}_{zz,\alpha}) + \sqrt{5}\big(\overline{\tau}_{xz,\alpha}\partial_x z_{\alpha\pm 1/2} - \rho K_{w,\alpha\pm 1/2}\big) \end{pmatrix},$$

where $(\partial_x \mathbf{D}_{\tau,\alpha})$ corresponds to diffusive terms, with second order derivatives of the velocity components in the case of a viscous stress tensor (see Sect. 3.2). $\mathbf{G}_{\tau,\alpha}^\pm$ is the momentum transference terms between the layers related to the stress tensor approximation. Actually, one may observe the similar structure of vectors $\mathbf{G}_{\tau,\alpha}^\pm$ and $\mathbf{G}_\alpha^\pm$, corresponding the latter to the momentum transference associated to convective terms.

Concerning non-hydrostatic contributions, as stated in [15], $\nabla_{\mathrm{NH}}\mathbf{Q}_\alpha$ and $\nabla_{\mathrm{NH}}\cdot\mathbf{X}_\alpha$ are the following operators:

$$\nabla_{\mathrm{NH}}\mathbf{Q}_\alpha = \begin{pmatrix} \partial_x(h_\alpha \overline{q}_\alpha) - (\delta(q\partial_x z))_\alpha \\ \frac{\sqrt{3}}{5}\left[\partial_x\left(h_\alpha\left(\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right)\right) + 10\overline{q}_\alpha\partial_x z_\alpha + \left(\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right)\partial_x h_\alpha - 10\widehat{(q\partial_x z)}_\alpha\right] \\ (\delta q)_\alpha \\ 2\sqrt{3}(\widehat{q}_\alpha - \overline{q}_\alpha) \\ \frac{2\sqrt{5}}{5}\big((\delta q)_\alpha - \pi_\alpha\big) \end{pmatrix},$$

for $\alpha \in \{1, \ldots, L\}$,

$$\nabla_{\mathrm{NH}}\cdot\mathbf{X}_\alpha = \begin{pmatrix} h_\alpha\partial_x\overline{u}_\alpha + 2\sqrt{3}\Phi_\alpha - 2\sqrt{3}\Lambda_\alpha\partial_x z_\alpha \\ \hline \overline{w}_\alpha - \overline{w}_{\alpha-1} - (\overline{u}_\alpha - \overline{u}_{\alpha-1})\partial_x z_{\alpha-1/2} - \sqrt{3}(\Phi_\alpha + \Phi_{\alpha-1}) \\ +\frac{2\sqrt{5}}{5}(\Psi_\alpha - \Psi_{\alpha-1}) + \sqrt{3}(\Lambda_\alpha + \Lambda_{\alpha-1})\partial_x z_{\alpha-1/2} \\ +\frac{\sqrt{3}}{10}(\Lambda_\alpha\partial_x h_\alpha - h_\alpha\partial_x\Lambda_\alpha - \Lambda_{\alpha-1}\partial_x h_{\alpha-1} + h_{\alpha-1}\partial_x\Lambda_{\alpha-1}) \\ \hline \frac{1}{5}\left[2\sqrt{5}\Psi_\alpha + \frac{\sqrt{3}}{3}(h_\alpha\partial_x\Lambda_\alpha - \Lambda_\alpha\partial_x h_\alpha)\right] \end{pmatrix}$$

for $\alpha \in \{2, \ldots, L\}$ and

$$\nabla_{\mathrm{NH}}\cdot\mathbf{X}_1 = \begin{pmatrix} h_1\partial_x\overline{u}_1 + 2\sqrt{3}\Phi_1 - 2\sqrt{3}\Lambda_1\partial_x z_1 \\ \hline \overline{w}_1 - \overline{u}_1\partial_x z_b + \sqrt{3}(\Lambda_1\partial_x z_b - \Phi_1) + \frac{\sqrt{3}}{10}(\Lambda_1\partial_x h_1 - h_1\partial_x\Lambda_1) + \frac{2\sqrt{5}}{5}\Psi_1 \\ \hline \frac{2\sqrt{5}}{5}\Psi_1 + \frac{\sqrt{3}}{15}(h_1\partial_x\Lambda_1 - \Lambda_1\partial_x h_1) \end{pmatrix},$$

These definitions of the NH-gradient and the NH-divergence operators satisfy the duality relation

$$\sum_{\alpha=1}^{L} \mathbf{X}_\alpha \cdot (\nabla_{\rm NH}\mathbf{Q}_\alpha) = -\sum_{\alpha=1}^{L} \mathbf{Q}_\alpha \cdot (\nabla_{\rm NH} \cdot \mathbf{X}_\alpha) + \partial_x\left(\sum_{\alpha=1}^{L} h_\alpha \left[\overline{q}_\alpha \overline{u}_\alpha + \frac{\sqrt{3}}{5}\Lambda_\alpha\left(\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right)\right]\right). \tag{38}$$

Notice that LIN-NH$_1$-STRESS model may be also written in a compact form. For the vectors of unknowns $\mathbf{X}_\alpha$ and $\mathbf{Q}_\alpha$, and the vectors related to the stress tensor, $\mathbf{D}_{\tau,\alpha}$ and $\mathbf{G}^{\pm}_{\tau,\alpha}$, the last component is removed. The rest of vectors are defined as in [15] for LIN-NH$_1$ model.

Remark 2. Concerning the numerical approximation of non-hydrostatic layer-averaged models, in [17] authors introduced a specific method for LDNH models (see [23]), which consider a layerwise constant horizontal velocity to approximate Euler system. It consists of a projection method based on a duality relation, which is similar to (38). In principle, the method introduced there could be adapted to the LIN-NH$_k$-STRESS models, combining it with an appropriate discretization of the viscous terms. However, it deserves especial attention and will be addressed in the future.

5. Layer-averaged models based on asymptotic analysis

In this section we derive several layer-averaged models that approximate the Navier–Stokes system, for different orders of approximation in the shallow parameter ε. It is motivated by the application of these models to geophysical flows. In particular, we consider here the usual asymptotic hypothesis for dry granular flows (see *e.g.* [19, 30]), although any other could be chosen in principle.

We define the parameter $\varepsilon = \mathcal{H}/L$, where $\mathcal{H}$ and L are the characteristic height and length of the domain, respectively. The shallow water hypothesis considers the ratio $\varepsilon \ll 1$, which implies that vertical variations are less important than horizontal ones. Denoting by ρ_0 and U the characteristic density and velocity, and with tildes ($\widetilde{\ }$) the non-dimensional variables, we consider

$$(x,z,t) = \big(L\widetilde{x}, \mathcal{H}\widetilde{z}, (L/U)\widetilde{t}\big), \qquad H = \mathcal{H}\widetilde{H}, \qquad \rho = \rho_0\widetilde{\rho}, \qquad (u,w) = (U\widetilde{u}, \varepsilon U\widetilde{w}),$$
$$p = \rho_0 U^2 \widetilde{p}, \qquad (\tau_{xx}, \tau_{xz}, \tau_{zz}) = \rho_0 U^2 (\varepsilon\widetilde{\tau_{xx}}, \widetilde{\tau_{xz}}, \varepsilon\widetilde{\tau_{zz}}).$$

Furthermore, by defining the Froude number $Fr = U^2/\sqrt{|g_z|\mathcal{H}}$ ($Fr = U^2/\sqrt{g\mathcal{H}}$ in Cartesian and $Fr = U^2/\sqrt{g\cos\theta\mathcal{H}}$ in local coordinates), the pressure is decomposed as

$$\widetilde{p} = \widetilde{\rho}\left(\frac{1}{Fr^2}\left(\widetilde{b} + \widetilde{H} - \widetilde{z}\right) + \varepsilon\widetilde{q}\right),$$

where the non-hydrostatic counterpart is supposed to be small with respect to the hydrostatic one. Finally, we assume a flow regime where $|g_x|/\big(Fr^2|g_z|\big) \sim \mathcal{O}(1)$.

The non-dimensional form of the Navier–Stokes system (1) reads (tildes are dropped for the sake of simplicity)

$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \varepsilon\big(\partial_t u + \partial_x\big(u^2\big) + \partial_z(u\,w)\big) + \frac{\varepsilon}{Fr^2}\partial_x(b+H) + \varepsilon^2\partial_x q = \frac{g_x}{Fr^2|g_z|} + \frac{1}{\rho}\big(\varepsilon^2\partial_x\tau_{xx} + \partial_z\tau_{xz}\big), \\ \varepsilon^2\big(\partial_t w + \partial_x(u\,w) + \partial_z\big(w^2\big)\big) + \varepsilon\partial_z q = \frac{\varepsilon}{\rho}(\partial_x\tau_{xz} + \partial_z\tau_{zz}), \end{cases} \tag{39a}$$

where the hydrostatic pressure has been cancelled with the gravitational force in the vertical momentum equation. Now, the layer-averaging approach is applied to the previous system, as it is done in Section 4, and the following non-dimensional layer-averaged system is reached:

$$\partial_t H + \partial_x(H\bar{\bar{u}}) = 0, \tag{40a}$$

$$\varepsilon\left[\partial_t(h_\alpha\overline{u}_\alpha)+\partial_x\left(h_\alpha\overline{u}_\alpha^2+\frac{h_\alpha^3\lambda_\alpha^2}{12}\right)\right]+\frac{\varepsilon}{Fr^2}h_\alpha\partial_x(b+H)+\varepsilon^2\partial_x(h_\alpha\overline{q}_\alpha)=\frac{g_x h_\alpha}{Fr^2|g_z|}$$
$$+\varepsilon^2\big(q_{\alpha+1/2}\partial_x z_{\alpha+1/2}-q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\big)+\varepsilon^2\partial_x\left(h_\alpha\frac{\overline{\tau}_{xx,\alpha}}{\rho}\right)$$
$$+\varepsilon\big(K_{\varepsilon,\alpha-1/2}-K_{\varepsilon,\alpha+1/2}\big)+\varepsilon\big(\widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2}-\widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2}\big),\tag{40b}$$

$$\varepsilon\partial_t\left(\frac{h_\alpha^2\lambda_\alpha}{12}\right)+\partial_x\left[\varepsilon\frac{h_\alpha^2\lambda_\alpha\overline{u}_\alpha}{12}+\varepsilon^2\left(\frac{h_\alpha\big(q_{\alpha+1/2}-q_{\alpha-1/2}\big)}{20}+\frac{h_\alpha\pi_\alpha}{30}\right)\right]+\varepsilon\frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{u}_\alpha+\varepsilon^2\overline{q}_\alpha\partial_x z_\alpha$$
$$=-\varepsilon^2\left(\frac{(q_{\alpha+1/2}-q_{\alpha-1/2})}{20}+\frac{\pi_\alpha}{30}\right)\partial_x h_\alpha+\frac{\varepsilon^2}{2}\big(q_{\alpha+1/2}\partial_x z_{\alpha+1/2}+q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\big)$$
$$+\varepsilon^2\partial_x\left(\frac{h_\alpha^2\zeta_{xx,\alpha}}{12\rho}\right)+\varepsilon^2\frac{h_\alpha\zeta_{xx,\alpha}}{12\rho}\partial_x h_\alpha+\frac{1}{\rho}\big(\varepsilon^2\overline{\tau}_{xx,\alpha}\partial_x z_\alpha-\overline{\tau}_{xz,\alpha}\big)-\frac{\varepsilon}{2}\big(K_{\varepsilon,\alpha+1/2}+K_{\varepsilon,\alpha-1/2}\big)$$
$$-\varepsilon\Gamma_{\alpha-1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}-\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha-1/2}}{2}\right)+\varepsilon\Gamma_{\alpha+1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}+\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha+1/2}}{2}\right),\tag{40c}$$

$$\varepsilon^2\left[\partial_t(h_\alpha\overline{w}_\alpha)+\partial_x\left(h_\alpha\overline{u}_\alpha\overline{w}_\alpha+\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{12}\right)\right]=\varepsilon\big(q_{\alpha-1/2}-q_{\alpha+1/2}\big)+\varepsilon\partial_x\left(h_\alpha\frac{\overline{\tau}_{xz,\alpha}}{\rho}\right)$$
$$+\varepsilon\big(K_{\varepsilon,w,\alpha-1/2}-K_{\varepsilon,w,\alpha+1/2}\big)+\varepsilon^2\big(\widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2}-\widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2}\big),\tag{40d}$$

$$\varepsilon^2\left[\partial_t\left(\frac{h_\alpha^2\varphi_\alpha}{12}\right)+\partial_x\left(\frac{h_\alpha^2\varphi_\alpha\overline{u}_\alpha}{12}+\frac{h_\alpha^4\lambda_\alpha\psi_\alpha}{360}\right)+\frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{w}_\alpha-\frac{h_\alpha^4\psi_\alpha^2}{720}+\frac{h_\alpha^3\lambda_\alpha\psi_\alpha}{360}\partial_x h_\alpha\right]$$
$$+\varepsilon\left(\frac{q_{\alpha+1/2}+q_{\alpha-1/2}}{2}-\overline{q}_\alpha\right)$$
$$=\varepsilon\left[\partial_x\left(\frac{h_\alpha^2\zeta_{xz,\alpha}}{12\rho}\right)+\frac{h_\alpha\zeta_{xz,\alpha}}{12\rho}\partial_x h_\alpha+\frac{1}{\rho}(\overline{\tau}_{xz,\alpha}\partial_x z_\alpha-\overline{\tau}_{zz,\alpha})\right]-\frac{\varepsilon}{2}\big(K_{\varepsilon,w,\alpha-1/2}+K_{\varepsilon,w,\alpha+1/2}\big)$$
$$-\varepsilon^2\Gamma_{\alpha-1/2}\left(\frac{h_\alpha\varphi_\alpha}{12}-\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{2}\right)+\varepsilon^2\Gamma_{\alpha+1/2}\left(\frac{h_\alpha\varphi_\alpha}{12}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{2}\right),\tag{40e}$$

$$\varepsilon^2\left[\partial_t\left(\frac{h_\alpha^3\psi_\alpha}{720}\right)+\partial_x\left(\frac{h_\alpha^3\psi_\alpha\overline{u}_\alpha}{720}+\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{360}\right)-\frac{h_\alpha^2\lambda_\alpha\varphi_\alpha}{120}\partial_x h_\alpha+\frac{h_\alpha^3\psi_\alpha\varphi_\alpha}{240}\right]+\varepsilon\frac{(q_{\alpha+1/2}-q_{\alpha-1/2})-\pi_\alpha}{30}$$
$$=\varepsilon\partial_x\left(\frac{h_\alpha^3\xi_{xz,\alpha}}{720\rho}\right)+\varepsilon\frac{h_\alpha^2\xi_{xz,\alpha}}{360\rho}\partial_x h_\alpha+\varepsilon\frac{h_\alpha}{12\rho}(\zeta_{xz,\alpha}\partial_x z_\alpha-\zeta_{zz,\alpha})+\varepsilon\frac{\overline{\tau}_{xz,\alpha}}{12\rho}\partial_x h_\alpha$$
$$-\frac{\varepsilon}{12}\big(K_{\varepsilon,w,\alpha+1/2}-K_{\varepsilon,w,\alpha-1/2}\big)-\varepsilon^2\Gamma_{\alpha-1/2}\left(\frac{h_\alpha^2\psi_\alpha}{360}-\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{12}\right)$$
$$+\varepsilon^2\Gamma_{\alpha+1/2}\left(\frac{h_\alpha^2\psi_\alpha}{360}+\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{12}\right),\tag{40f}$$

for $\alpha=1,\dots,L$, combined with the set constraints (31) written in non-dimensional form. In the previous system, the viscous terms at the interfaces are

$$K_{\varepsilon,\alpha+1/2}=\frac{1}{\rho}\left(\varepsilon\frac{\overline{\tau}_{xx,\alpha}+\overline{\tau}_{xx,\alpha+1}}{2}\partial_x z_{\alpha+1/2}-\frac{1}{\varepsilon}\frac{\overline{\tau}_{xz,\alpha}+\overline{\tau}_{xz,\alpha+1}}{2}\right),$$

$$K_{\varepsilon,w,\alpha+1/2}=\frac{1}{\rho}\left(\frac{\overline{\tau}_{xz,\alpha}+\overline{\tau}_{xz,\alpha+1}}{2}\partial_x z_{\alpha+1/2}-\frac{\overline{\tau}_{zz,\alpha}+\overline{\tau}_{zz,\alpha+1}}{2}\right).$$

In the next, we see how several layer-averaged models approximating system (1) are deduced, depending on the order of approximation in ε that is considered in the layer-averaged system (40). Let us notice that all the models presented in the next subsections satisfy an energy balance.

5.1. Fully non-hydrostatic pressure systems

When considering the full system without simplifications, we obtain the models presented in Section 4. Such systems are fully non-hydrostatic models, and they are the more general cases in the framework of the asymptotic analysis. Thus, when no simplifications are made in system (40), we recover system LIN-NH$_2$-STRESS ((30), (31)) once we come back to dimensional variables. Similarly, we could perform the layerwise lineal vertical velocity case in the layer-averaging procedure and we would obtain system LIN-NH$_1$-STRESS ((35), (36)).

5.2. Systems with viscous dependent pressure: LIN-H-STRESS

Here we see the following level in the asymptotic analysis to obtain a model that approximates the Navier–Stokes system with a dissipative energy balance. It consists of neglecting terms up to order ε^2 in the vertical momentum equation in (40),

Thus, from the equations related to the vertical velocity variables ($\overline{w}_\alpha, \varphi_\alpha\psi_\alpha$) in (40), we obtain (in dimensional variables)

$$\begin{aligned}
q_{\alpha+1/2} - q_{\alpha-1/2} &= \partial_x\left(h_\alpha \frac{\overline{\tau}_{xz,\alpha}}{\rho}\right) + K_{w,\alpha-1/2} - K_{w,\alpha+1/2},\\
\frac{q_{\alpha+1/2}+q_{\alpha-1/2}}{2} - \overline{q}_\alpha &= \partial_x\left(\frac{h_\alpha^2\zeta_{xz,\alpha}}{12\rho}\right) + \frac{h_\alpha\zeta_{xz,\alpha}}{12\rho}\partial_x h_\alpha + \frac{1}{\rho}(\overline{\tau}_{xz,\alpha}\partial_x z_\alpha - \overline{\tau}_{zz,\alpha}) - \frac{1}{2}\big(K_{w,\alpha-1/2}+K_{w,\alpha+1/2}\big),\\
\frac{\big(q_{\alpha+1/2}-q_{\alpha-1/2}\big)-\pi_\alpha}{30} &= \partial_x\left(\frac{h_\alpha^3\xi_{xz,\alpha}}{720\rho}\right) + \frac{h_\alpha^2\xi_{xz,\alpha}}{360\rho}\partial_x h_\alpha + \frac{h_\alpha}{12\rho}(\zeta_{xz,\alpha}\partial_x z_\alpha - \zeta_{zz,\alpha}) + \frac{\overline{\tau}_{xz,\alpha}}{12\rho}\partial_x h_\alpha\\
&\quad - \frac{1}{12}\big(K_{w,\alpha+1/2}-K_{w,\alpha-1/2}\big).
\end{aligned} \tag{41}$$

These expressions allow us to write the non-hydrostatic terms in the model in terms of the stress tensor components. From the first equation, taking into account that $q_{N+1/2}=0$, we find the value of $q_{\alpha-1/2}$, for $\alpha = L, L-1, \ldots, 1$, which are used to obtain

$$\overline{q}_\alpha = -\left(\sum_{\beta=\alpha+1}^{N}\partial_x\left(\frac{h_\beta\overline{\tau}_{xz,\beta}}{\rho}\right) + \partial_x\left(\frac{h_\alpha\overline{\tau}_{xz,\alpha}}{2\rho}\right) + \partial_x\left(\frac{h_\alpha^2\zeta_{xz,\alpha}}{12\rho}\right) + \frac{h_\alpha\zeta_{xz,\alpha}}{12\rho}\partial_x h_\alpha + \overline{\tau}_{xz,\alpha}\partial_x z_\alpha - \overline{\tau}_{zz,\alpha}\right).$$

Then, it is used in the second equation in (40) to write the non-hydrostatic terms (on the right-hand side) as

$$\begin{aligned}
Q_u &= -\partial_x(h_\alpha\overline{q}_\alpha) + q_{\alpha+1/2}\partial_x z_{\alpha+1/2} - q_{\alpha-1/2}\partial_x z_{\alpha-1/2} = \partial_{xx}\left(\frac{h_\alpha^3\zeta_{xz,\alpha}}{12\rho}\right) + z_{\alpha+1/2}\sum_{\beta=\alpha+1}^{N}\partial_{xx}\left(\frac{h_\beta\overline{\tau}_{xz,\beta}}{\rho}\right)\\
&\quad - z_{\alpha-1/2}\sum_{\beta=\alpha}^{N}\partial_{xx}\left(\frac{h_\beta\overline{\tau}_{xz,\beta}}{\rho}\right) + \partial_{xx}\left(\frac{h_\alpha\overline{\tau}_{xz,\alpha}}{\rho}z_\alpha\right) - \partial_x\left(\frac{h_\alpha\overline{\tau}_{zz,\alpha}}{\rho}\right) + K_{w,\alpha-1/2}\partial_x z_{\alpha-1/2}\\
&\quad - K_{w,\alpha+1/2}\partial_x z_{\alpha+1/2}.
\end{aligned} \tag{42}$$

Moreover, defining

$$Q_{\lambda,1} = \frac{(q_{\alpha+1/2}-q_{\alpha-1/2})}{20} + \frac{\pi_\alpha}{30} = -\partial_x\left(\frac{h_\alpha^3\xi_{xz,\alpha}}{720\rho}\right) - \frac{h_\alpha^2\xi_{xz,\alpha}}{360\rho}\partial_x h_\alpha - \frac{h_\alpha}{12\rho}(\zeta_{xz,\alpha}\partial_x z_\alpha - \zeta_{zz,\alpha}) + h_\alpha\partial_x\left(\frac{\overline{\tau}_{xz,\alpha}}{12\rho}\right),$$

and

$$Q_{\lambda,2} = -\overline{q}_\alpha\partial_x z_\alpha + \frac{1}{2}\big(q_{\alpha+1/2}\partial_x z_{\alpha+1/2} + q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\big)$$

$$= \partial_x\left(\frac{h_\alpha\overline{\tau}_{xz,\alpha}}{4\rho}\right)\partial_x h_\alpha + \left(\partial_x\left(\frac{h_\alpha^2\zeta_{xz,\alpha}}{12\rho}\right) + \frac{h_\alpha\zeta_{xz,\alpha}}{12\rho}\partial_x h_\alpha + \frac{1}{\rho}(\overline{\tau}_{xz,\alpha}\partial_x z_\alpha - \overline{\tau}_{zz,\alpha})\right)\partial_x z_\alpha$$
$$- \frac{1}{2}\big(K_{w,\alpha+1/2}\partial_x z_{\alpha+1/2} + K_{w,\alpha-1/2}\partial_x z_{\alpha-1/2}\big),$$

the non-hydrostatic contribution in the third equation in (on the right-hand side) is

$$-\partial_x(h_\alpha Q_{\lambda,1}) - Q_{\lambda,1}\partial_x h_\alpha + Q_{\lambda,2}.$$

Thus, we get the system LIN-H-STRESS, which reads (in dimensional form)

$$\partial_t H + \partial_x(H\bar{\bar{u}}) = 0, \tag{43a}$$

$$\partial_t(h_\alpha\overline{u}_\alpha) + \partial_x\left(h_\alpha\overline{u}_\alpha^2 + \frac{h_\alpha^3\lambda_\alpha^2}{12}\right) + |g_z|h_\alpha\partial_x(z_b + H) = Q_u + \partial_x\left(h_\alpha\frac{\overline{\tau}_{xx,\alpha}}{\rho}\right) + K_{\alpha-1/2} - K_{\alpha+1/2}$$
$$+ \widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2}, \tag{43b}$$

$$\partial_t\left(\frac{h_\alpha^2\lambda_\alpha}{12}\right) + \partial_x\left(\frac{h_\alpha^2\lambda_\alpha\overline{u}_\alpha}{12}\right) + \frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{u}_\alpha = -\partial_x(h_\alpha Q_{\lambda,1}) - Q_{\lambda,1}\partial_x h_\alpha + Q_{\lambda,2} + \partial_x\left(\frac{h_\alpha^2\zeta_{xx,\alpha}}{12\rho}\right) + \frac{h_\alpha\zeta_{xx,\alpha}}{12\rho}\partial_x h_\alpha$$
$$+ \frac{\overline{\tau}_{xx,\alpha}}{\rho}\partial_x z_\alpha - \frac{\overline{\tau}_{xz,\alpha}}{\rho} - \frac{1}{2}\big(K_{\alpha+1/2} + K_{\alpha-1/2}\big) - \Gamma_{\alpha-1/2}\left(\frac{h_\alpha\lambda_\alpha}{12} - \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha-1/2}}{2}\right)$$
$$+ \Gamma_{\alpha+1/2}\left(\frac{h_\alpha\lambda_\alpha}{12} + \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha+1/2}}{2}\right), \tag{43c}$$

for $\alpha = 1, \ldots, N$. Notice that the vertical velocity is not an unknown of the model, therefore, it is computed from the incompressibility condition as

$$w_\alpha(z) = \overline{w}_\alpha + (-\partial_x\overline{u}_\alpha + \lambda_\alpha\partial_x z_\alpha)(z - z_\alpha) - \partial_x\lambda_\alpha\left(\frac{(z - z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right), \tag{44}$$

where

$$\overline{w}_\alpha = \overline{u}_\alpha\partial_x z_\alpha - \sum_{\beta=1}^{\alpha-1}\partial_x(h_\beta\overline{u}_\beta) - \frac{\partial_x(h_\alpha\overline{u}_\alpha)}{2} + \partial_x\left(\frac{h_\alpha^2\lambda_\alpha}{8}\right) - \frac{h_\alpha^2}{24}\partial_x\lambda_\alpha.$$

Let us remark that it is no more than the layer-integrated incompressibility condition together with the profile of $w_\alpha(z)$ (10) and restrictions (12).

It is worth noticing that we have obtained system (43) by means of some simplifications, related to the shallowness parameter (ε), in our layer-averaged LIN-NH$_2$-STRESS model. However, it has not been proven that this model (43) is a layer-averaged approximation of the corresponding continuous model with viscous dependent pressure. Actually, it holds. Let us detail it. Consider now model (39) and keep first-order terms in the vertical momentum equation (39a), therefore neglecting the vertical acceleration. That is, $D_t w$, the material derivative of the vertical velocity, is neglected. In that case, we get the system

$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \varepsilon\big(\partial_t u + \partial_x(u^2) + \partial_z(u\,w)\big) + \dfrac{\varepsilon}{Fr^2}\partial_x(b + h) + \varepsilon^2\partial_x q = \dfrac{g_x}{Fr^2|g_z|} + \dfrac{1}{\rho}\big(\varepsilon^2\partial_x\tau_{xx} + \partial_z\tau_{xz}\big), & (45a) \\ \varepsilon\partial_z q = \dfrac{\varepsilon}{\rho}(\partial_x\tau_{xz} + \partial_z\tau_{zz}). & (45b) \end{cases}$$

Now, equation (45b) is integrated in z and its result is put in the horizontal momentum equation (45a). It gives

$$\varepsilon\big(\partial_t u + \partial_x(u^2) + \partial_z(u\,w)\big) + \frac{\varepsilon}{Fr^2}\partial_x(b + h) = \frac{g_x}{Fr^2|g_z|} + \varepsilon^2\partial_x\tau_{xx} + \partial_z\tau_{xz} + \varepsilon^2\partial_x\left(\int_z^{b+h}(\partial_x\tau_{xz} + \partial_z\tau_{zz})\,\mathrm{d}z\right).$$

The following result is obtained, whose proof, that is analogous to computations in Section 5 in [8], is omitted with the purpose of brevity.

Theorem 3. *System LIN-H-STRESS (43) is a layer-averaged discretization with layerwise linear horizontal and parabolic vertical velocities ($u_\alpha \in \mathbb{P}_1$, $w_\alpha \in \mathbb{P}_2$) of the Navier–Stokes system with viscous dependent pressure given by*

$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \partial_t u + \partial_x(u^2) + \partial_z(u\,w) + |g_z|\partial_x(z_b+h) = \frac{1}{\rho}\Big(\partial_x\tau_{xx} + \partial_z\tau_{xz} + \partial_x\Big(\int_z^{b+h}(\partial_x\tau_{xz} + \partial_z\tau_{zz})\,\mathrm{d}z\Big)\Big). \end{cases}$$

In addition, previous model satisfies the following energy balance:

Theorem 4. *Let us consider the LIN-H-STRESS model defined by (43) and definitions (41), (44), with the stress tensor components defined by (22), (24), (26), and the terms $K_{\alpha\pm1/2}, K_{w,\alpha\pm1/2}$ given by (18). The following energy balance is satisfied*

$$\begin{aligned} &\partial_t\left(\sum_{\alpha=1}^N E_\alpha\right) + \partial_x\left[\sum_{\alpha=1}^N\left(\overline{u}_\alpha\left(E_\alpha + gh_\alpha\frac{h}{2} + \frac{h_\alpha^3\lambda_\alpha^2}{12} + h_\alpha\overline{q}_\alpha\right) + \lambda_\alpha\left(\frac{h_\alpha^2\pi_\alpha}{30} + \frac{h_\alpha^2\left(q_{\alpha+1/2} - q_{\alpha-1/2}\right)}{20}\right)\right.\right. \\ &\quad\left.\left. - \frac{1}{\rho}\left(h_\alpha\overline{u}_\alpha\overline{\tau}_{xx,\alpha} + \frac{h_\alpha^3\lambda_\alpha\zeta_{xx,\alpha}}{12} + h_\alpha\overline{w}_\alpha\overline{\tau}_{xz,\alpha} + \frac{h_\alpha^3\varphi_\alpha\zeta_{xz,\alpha}}{12} + \frac{h_\alpha^5\psi_\alpha\xi_{xz,\alpha}}{720}\right)\right)\right] \\ &\quad\le -\frac{1}{\rho}\sum_{\alpha=1}^N\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\left[\tau_{xx,\alpha}(z)\left(\overline{[\partial_x u]} + \partial_x\lambda_\alpha(z-z_\alpha)\right) + \tau_{zz,\alpha}(z)\left(\overline{[\partial_z w]}_\alpha + \psi_\alpha(z-z_\alpha)\right)\right. \\ &\quad\quad\left. + \tau_{xz,\alpha}(z)\left(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]} + (\partial_x\varphi_\alpha - \psi_\alpha\partial_x z_\alpha)(z-z_\alpha) + \partial_x\psi_\alpha\left(\frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right)\right)\right]\mathrm{d}z \\ &\quad\quad - \frac{1}{\rho}\left(\beta_0 + \frac{\beta_1}{|\boldsymbol{U}|}\right)\left(1 + (\partial_x b)^2\right)^{3/2}\left(u_{1/2}^+\right)^2, \end{aligned}$$

where

$$E_\alpha \;:=\; h_\alpha\left(\frac{\overline{u}_\alpha^2}{2} + \frac{(h_\alpha\lambda_\alpha)^2}{24} + g\left(z_b + \frac{h}{2}\right)\right).$$

Proof. The proof is analogous to proof of Theorem 1. □

When considering a stress tensor proportional to the strain rate, as in Section 3.2, a dissipative energy balance is satisfied, analogously to Corollary 1.

In addition, let us remark that analogously to what is done in Section 4.3, a simplified version of the model presented in this section with linear vertical velocity ($w_\alpha \in \mathbb{P}_1$, $q_\alpha \in \mathbb{P}_2$) can be deduced.

5.3. System with hydrostatic pressure and vertical diffusion: LIN-H

In this section we consider the model that is obtained by neglecting terms up to order ε^2 in the horizontal momentum equation in (40). It is equivalent to consider a hydrostatic pressure

$$p = \rho\frac{1}{Fr^2}(b + H - z).$$

We obtain the hydrostatic LIN-H model

$$\partial_t H + \partial_x(H\bar{\bar{u}}) = 0, \tag{46a}$$

$$\partial_t(h_\alpha\overline{u}_\alpha) + \partial_x\left(h_\alpha\overline{u}_\alpha^2 + \frac{h_\alpha^3\lambda_\alpha^2}{12}\right) + |g_z|h_\alpha\partial_x(z_b+H) = K_{\alpha-1/2} - K_{\alpha+1/2} + \widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2}, \tag{46b}$$

$$\partial_t\left(\frac{h_\alpha^2\lambda_\alpha}{12}\right) + \partial_x\left(\frac{h_\alpha^2\lambda_\alpha\overline{u}_\alpha}{12}\right) + \frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{u}_\alpha = -\frac{\overline{\tau}_{xz,\alpha}}{\rho} - \frac{1}{2}\big(K_{\alpha+1/2}+K_{\alpha-1/2}\big) - \Gamma_{\alpha-1/2}\left(\frac{h_\alpha\lambda_\alpha}{12} - \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha-1/2}}{2}\right) + \Gamma_{\alpha+1/2}\left(\frac{h_\alpha\lambda_\alpha}{12} + \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha+1/2}}{2}\right), \tag{46c}$$

for $\alpha = 1,\dots,L$, where the vertical velocity is again defined from the incompressibility condition, taking the form (44), and the viscous terms at the interfaces are

$$K_{\alpha+1/2} = -\frac{1}{\rho}\frac{\overline{\tau}_{xz,\alpha}+\overline{\tau}_{xz,\alpha+1}}{2}, \qquad \text{and} \qquad K_{1/2} = -\frac{1}{\rho}\left(\beta_0 + \frac{\beta_1}{|\boldsymbol{U}|}\right)u_{1/2}^+. \tag{47}$$

In this case we also obtain that this model is a layer-averaged approximation of the Navier–Stokes system up to first order, where the horizontal momentum equation is

$$\varepsilon\big(\partial_t u + \partial_x(u^2) + \partial_z(u\,w)\big) + \varepsilon\frac{1}{Fr^2}\partial_x(b+H) = \frac{g_x}{Fr^2|g_z|} + \frac{1}{\rho}\partial_z\tau_{xz}.$$

Theorem 5. *System LIN-H (46) is a layer-averaged discretization with layerwise linear horizontal and parabolic vertical velocities ($u_\alpha \in \mathbb{P}_1$, $w_\alpha \in \mathbb{P}_2$) of the hydrostatic system*

$$\begin{cases}\partial_x u + \partial_z w = 0,\\ \partial_t u + \partial_x(u^2) + \partial_z(u\,w) + |g_z|\partial_x(z_b+H) = \frac{1}{\rho}\partial_z\tau_{xz}.\end{cases}$$

Proof. It is analogous to the proof of Theorem 3.

Let us develop the model for the case $\tau = \rho\nu D$ (see Sect. 3.2). The strain rate tensor is written in non-dimensional form (without tildes for simplicity) as

$$D(\boldsymbol{U}) = \frac{U}{\mathcal{H}}\,\frac{1}{2}\begin{pmatrix} 2\varepsilon\partial_x u & \partial_z u + \varepsilon^2\partial_x w\\ \partial_z u + \varepsilon^2\partial_x w & 2\varepsilon\partial_z w\end{pmatrix}.$$

Notice that it implies that

$$\tau_{xx} = \rho\,\nu\partial_x u, \qquad \tau_{xz} = \rho\,\frac{\nu}{2}\big(\partial_z u + \varepsilon^2\partial_x w\big), \qquad \tau_{zz} = \rho\,\nu\partial_z w.$$

Focusing on the LIN-H model, only the term τ_{xz} appears. By neglecting terms of order ε^2, we obtain

$$\tau_{xz} = \rho\,\frac{\nu}{2}\partial_z u,$$

which is approximated in the layer-averaged framework as

$$\tau_{xz,\alpha}(z) = \overline{\tau}_{xz,\alpha} + \zeta_{xz,\alpha}(z - z_\alpha),$$

with

$$\overline{\tau}_{xz,\alpha} = \frac{\nu_\alpha^0}{2}\overline{[\partial_z u]}_\alpha, \qquad \zeta_{xz,\alpha} = \frac{\nu_{xz,\alpha}^1}{2}\overline{[\partial_z u]}_\alpha, \tag{48}$$

and $\overline{[\partial_z u]}_\alpha$ given by (20). Note that the term $\zeta_{xz,\alpha}$ has no influence on this model and we only need the average $\overline{\tau}_{xz,\alpha}$ in the hydrostatic LIN-H model. □

This model also satisfies an energy balance:

Theorem 6. *Let us consider the LIN-H model defined by* (46)*, with the stress tensor components defined by* (47) *and* (48)*. The following dissipative energy balance is satisfied:*

$$\partial_t\left(\sum_{\alpha=1}^{N} E_\alpha\right)+\partial_x\left[\sum_{\alpha=1}^{N}\left(\overline{u}_\alpha\left(E_\alpha+|g_z|h_\alpha\frac{h}{2}+\frac{h_\alpha^3\lambda_\alpha^2}{12}\right)\right)\right]\leq-\frac{1}{\rho}\sum_{\alpha=1}^{N}\frac{\nu_{xz,\alpha}^0 h_\alpha}{2}\left(\overline{[\partial_z u]}_\alpha\right)^2\mathrm{d}z-\frac{1}{\rho}\left(\beta_0+\frac{\beta_1}{|\boldsymbol{U}|}\right)\left(u_{1/2}^+\right)^2$$

where

$$E_\alpha:=h_\alpha\left(\frac{\overline{u}_\alpha^2}{2}+\frac{(h_\alpha\lambda_\alpha)^2}{24}+|g_z|\left(z_b+\frac{h}{2}\right)\right).$$

Proof. The proof is analogous to proof of Theorem 1. □

5.4. Summary of models and relation with precedent models

For the sake of clarity, in this section we summarise all the models introduced in this work, as well as their relation with previous models proposed by our group and others. Actually, the models presented here are generalisations of plenty of previous models in the literature. That is, by making appropriate simplifications in a certain way over the assumed profiles for the velocity profile, one is able to recover all the family of layer-averaged models that are summarised below. An important remark is that, for all models obtained as particular (or simplified) cases of the LIN-NH$_2$-STRESS model, their energy balances (and respective proofs) are obtained making the same assumptions over the general proof of Theorem 1. That is, the energy balance for the more complete model, and its proof, reduce well to the energy balance of each simplified model.

The presented models are related to previous ones in the literature as follows:

(i) LIN-NH$_2$-STRESS model, presented in Section 5.1, is a generalisation of LIN-NH$_2$ model proposed in [15] from the Euler to Navier–Stokes case. Analogously for LIN-NH$_1$-STRESS with respect to LIN-NH$_1$.
Furthermore, if previous models are simplified by assuming a layerwise constant horizontal velocity ($u_\alpha \in \mathbb{P}_0$), the obtained models are generalisations of the models LDNH$_k$ ($k = 0, 2$ depending on the choice for the profile of w_α, constant or linear) introduced in [23] toward the Navier–Stokes case. That is, LIN-NH$_{1,2}$-STRESS are generalisations of the models LDNH$_{0,2}$ [23] to the Navier–Stokes case and layerwise linear horizontal velocity.

(ii) LIN-H-STRESS model (43), presented in Section 5.2, is a generalisation of the model introduced in [8] to the layerwise linear horizontal velocity, and also layerwise linear viscosity. Note that, in [8], both horizontal and vertical velocities are supposed to be layerwise constant functions. Then, assuming that $\lambda_\alpha = 0$, removing last equation in system LIN-H-STRESS (43) and taking $\tau_{xz,\alpha}(z) = \overline{\tau}_{xz,\alpha}$ layerwise constant ($u_\alpha, w_\alpha \in \mathbb{P}_0$), we recover the layer-averaged model proposed in [8].
Moreover, we can consider $u_\alpha \in \mathbb{P}_0$ as in [8] but $w_\alpha \in \mathbb{P}_1$ and the resulting model is also an extension of the model in that work to the case of non-layerwise constant shear $\tau_{xz,\alpha}$. In particular, the difference between considering w_α layerwise linear or constant is the fact of including the term involving $\zeta_{xz,\alpha}$ in Q_u (42).
Notice that the methodology employed in that work is different from the one considered here. We first apply the layer-averaged discretization to system (45), and later use the definitions of $\overline{q}_\alpha, q_{\alpha+1/2}, \pi_\alpha$ to write non-hydrostatic contributions in terms of the viscous terms. However, in [8] authors first integrate the vertical momentum equation in (45) to obtain the non-hydrostatic pressure q, and then it is replaced in the horizontal momentum equation. Finally, the layer-averaged discretization is applied to the resulting horizontal momentum equation. Here, our result (Thm. 3) proves that these are indeed equivalent procedures.

(iii) LIN-H model (46), presented in Section 5.3, is the generalisation of the model used in [18], to the case of layerwise linear horizontal velocity.

Table 1. Summary of models introduced in this work and in [15], discrete spaces for $u_\alpha(z), w_\alpha(z), q_\alpha(z)$, unknowns, maximum degree of the derivatives appearing in each model and the original model that is approximated.

Model	Disc. spaces (u_α, w_α, q_α)	Dimension	Unknowns	Max. degree of derivatives	Approximated model
LIN-NH$_2$-STRESS	($\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$)	8L+1	$H, \{\overline{u}_\alpha, \lambda_\alpha\}, \{\overline{w}_\alpha, \varphi_\alpha, \psi_\alpha\}$ $\{\overline{q}_\alpha, q_{\alpha-1/2}, \pi_\alpha\}$	2	Navier–Stokes
LIN-NH$_1$-STRESS	($\mathbb{P}_1$, $\mathbb{P}_1$, $\mathbb{P}_2$)	6L+1	$H, \{\overline{u}_\alpha, \lambda_\alpha\}, \{\overline{w}_\alpha, \varphi_\alpha\}$ $\{\overline{q}_\alpha, q_{\alpha-1/2}\}$	2	Navier–Stokes
LIN-H-STRESS	($\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$)	2L+1	$H, \{\overline{u}_\alpha, \lambda_\alpha\}$	4	Navier–Stokes ($D_t w$ neglected)
LIN-NH$_2$	($\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$)	8L+1	$H, \{\overline{u}_\alpha, \lambda_\alpha\}, \{\overline{w}_\alpha, \varphi_\alpha, \psi_\alpha\}$ $\{\overline{q}_\alpha, q_{\alpha-1/2}, \pi_\alpha\}$	1	Euler
LIN-NH$_1$	($\mathbb{P}_1$, $\mathbb{P}_1$, $\mathbb{P}_2$)	6L+1	$H, \{\overline{u}_\alpha, \lambda_\alpha\}, \{\overline{w}_\alpha, \varphi_\alpha\}$ $\{\overline{q}_\alpha, q_{\alpha-1/2}\}$	1	Euler

Notice that LIN-NH$_2$-STRESS and LIN-H-STRESS models use the same spaces of approximation for the variables, but the LIN-H-STRESS model neglects the vertical acceleration. It allows us to write the pressure unknowns in terms of the stress tensor components, and therefore in terms of $\{\overline{u}_\alpha, \lambda_\alpha\}$. They are collected in the terms $Q_u, Q_{\lambda,1}, Q_{\lambda,2}$. Thus, the unknowns of the LIN-H-STRESS model are only the total height and the ones related to the horizontal velocity profile, but it involves high order derivatives of the variables. For instance, in term Q_u it appears the term $\partial_{xx}\overline{\tau}_{xz,\alpha}$, which involves (among others) the term $\partial_x \overline{w}_\alpha$, being this one computed in terms of $\partial_x \overline{u}_\alpha$. As conclusion, fourth-order space derivatives appear in that model.

In Table 1 the main characteristics of the models introduced in this work and in [15] are summarized.

6. Numerical examples for uniform geophysical flows

In this section, we illustrate the advantage of using the proposed layer-averaged model with layerwise linear horizontal velocity. In particular, we consider some geophysical flows with constant (Newtonian fluids) and variable (viscoplastic fluids) viscosity in a uniform configuration over an inclined slope ($\theta > 0$). We choose this regime because we want to evaluate the vertical approximation, so we neglect horizontal variations, and then it is possible to obtain analytical solutions for the velocity and shear stress.

Our goal is showing how the proposed approach (layerwise linear horizontal velocity) allows us to significantly improve, with respect to the layerwise constant velocity approach, the approximation of the analytical vertical profiles of the uniform flow solution. The design of an efficient numerical scheme for the full non-hydrostatic model is a key but hard problem, which in particular is necessary to make these models usable from the practical point of view. Then, due to the difficulty of such schemes, they deserve special attention (see [17]) and it is out of the scope of this work.

The procedure to obtain the analytical solution is similar for all the cases considered. We write the Navier–Stokes system (1) in local coordinates, and the uniform flow assumption leads to

$$\partial_z p = -\rho g \cos\theta, \qquad \partial_z \tau_{xz} = \rho g \sin\theta,$$

with boundary conditions $p_{|_{z=H}} = \tau_{xz|_{z=H}} = 0$, where H denotes the height of the fluid. Let us consider $\rho = 1$ for simplicity. The integration of the first condition gives us the hydrostatic regime

$$p(z) = g\cos\theta(H - z).$$

The second one implies that the stress tensor, defined as in Section 3.2, is

$$\tau_{xz}(z) = \left(\frac{\nu}{2}\partial_z u\right)(z) = g\sin\theta(H - z), \tag{49}$$

where ν is again the kinematic viscosity coefficient that must be defined by the particular rheology in each case. Note that for this flow configuration the shear stress is a linear function, so we can correctly assess the accuracy of the proposed approach and the correction in Section 4.2. Let us also remark that (49) holds for all the considered flows.

Let us denote by $u_{\text{an}}(z)$ is the analytical velocity profile for the considered uniform flow. Then, define

$$\overline{u}_\alpha = \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_{\text{an}}(z)\,\mathrm{d}z, \qquad \text{and} \qquad \lambda_\alpha = (\partial_z u_{\text{an}}(z))_{|z=z_\alpha},$$

so we have

$$u_\alpha(z) = \overline{u}_\alpha + \lambda_\alpha(z - z_\alpha).$$

The viscosity (subscripts xz are neglected for the sake of clarity) is given by

$$\nu_\alpha(z) = \nu_\alpha^0 + \nu_\alpha^1(z - z_\alpha)$$

with

$$\nu_\alpha^0 = \nu_{\text{an}|z=z_\alpha}, \qquad \text{and} \qquad \nu_\alpha^1 = (\partial_z \nu_{\text{an}}(z))_{|z=z_\alpha},$$

being ν_{an} the corresponding viscosity depending on the chosen rheology. The shear stress $\tau_{xz,\alpha}$, whose analytical expression ($\tau_{xz,\text{an}}$) is (49), will be computed following (23) and (24) as

$$\tau_{xz,\alpha}(z) = \overline{\tau}_{xz,\alpha} + \zeta_{xz,\alpha}(z - z_\alpha), \tag{50}$$

with

$$\overline{\tau}_{xz,\alpha} = \frac{\nu_\alpha^0}{2}\overline{[\partial_z u]}_\alpha \qquad \text{and} \qquad \zeta_{xz,\alpha} = \frac{\nu_\alpha^1}{2}\overline{[\partial_z u]}_\alpha,$$

whereas for the second-order correction of this term, $\widetilde{\tau}_{xz,\alpha}$, we need to replace $\zeta_{xz,\alpha}$ in previous equation by

$$\widetilde{\zeta}_{xz,\alpha} = \frac{\nu_\alpha^1}{2}\overline{[\partial_z u]}_\alpha + \frac{\nu_\alpha^0}{2}\widetilde{\chi}_\alpha,$$

for $\widetilde{\chi}_\alpha$ defined by (33b).

In the following, we show the accuracy of the proposed approach for three different rheologies. We start by a simple Newtonian fluid and later two viscoplastic fluids.

6.1. Newtonian fluids

Let us consider a Newtonian fluid, for which $\nu_{\text{an}} = \nu_0$ constant. From (49) we get

$$\partial_z u_{\text{an}}(z) = \frac{2g\sin\theta}{\nu_0}(H - z), \qquad u_{\text{an}}(z) = \frac{g\sin\theta}{\nu_0}\Big(H^2 - (H - z)^2\Big).$$

Let us consider a flow with height $H = 1$, viscosity $\nu_{\text{an}} = 10^{-2}$ and a slope $\theta = 5°$. Figure 2 shows the comparison between the analytical solution and the layerwise approximations with only 3 vertical layers for the velocity, its vertical derivative and the shear stress τ_{xz}. We also show the results of the previous layer-averaged model with layerwise constant velocity. In Figure 2a we see that the linear approach allows us to have a more accurate approximation of the velocity profile, even with a few layers. In Figure 2b we can observe the profile of $\partial_z u$, that is layerwise constant. Special attention deserves Figure 2c, where the shear stress τ_{xz} is shown. Here, we see the key role of the second-order correction of the shear stress $\widetilde{\tau}_{xz,\alpha}$. Actually, we can see that the approximation without the correction matches with the approximation by a layerwise constant velocity. It is due to the fact that the slope $\zeta_{xx,\alpha}$ in (50) is zero in this case, since $\nu = \nu_0$ constant. However, it is not the case for $\widetilde{\tau}_{xz,\alpha}$ where now we recover the correct slope of $\tau_{xz,\text{an}}$. Concretely, we see that the lines corresponding to $\widetilde{\tau}_{xz,\alpha}$ and $\tau_{xz,\text{an}}$ overlap, as expected.

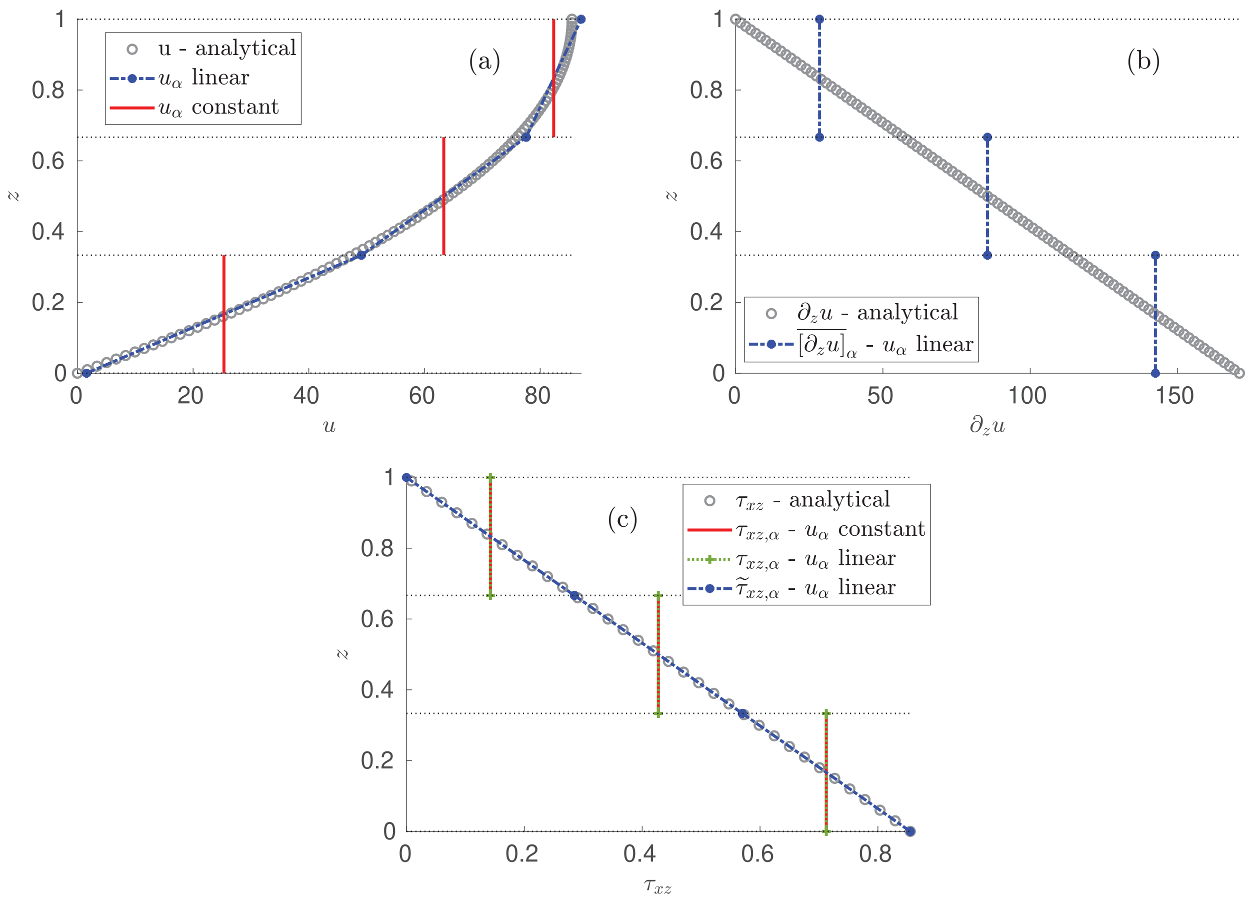

FIGURE 2. Comparison between the analytical vertical profiles (grey circles) and layerwise approximations with constant (solid red lines) and linear (dot-dashed blue and dotted green lines) horizontal velocity, and 3 layers: (a) Horizontal velocity (u); (b) vertical derivative ($\partial_z u$); (c) stress tensor component τ_{xz}, where $\widetilde{\tau}_{xz,\alpha}$ denotes the second-order correction of $\tau_{xz,\alpha}$ (33).

6.2. Dry granular flows

For the case of dry granular flows, where the $\mu(I)$-rheology (see [32]) is considered, the stress tensor is only defined if $|\partial_z u| > 0$, otherwise the tensor is multivalued. In order to deal with this difficulty, a well-known technique is to use a regularised viscosity coefficient (see *e.g.* [18, 20]), with a small regularisation parameter $\delta > 0$. In practice, we set $\delta = 10^{-5}$.

In the case of granular flows, we consider the viscosity

$$\nu_{\text{an}}(z) = \frac{g \sin\theta (H - z)}{\sqrt{|\partial_z u|^2/4 + \delta^2}},$$

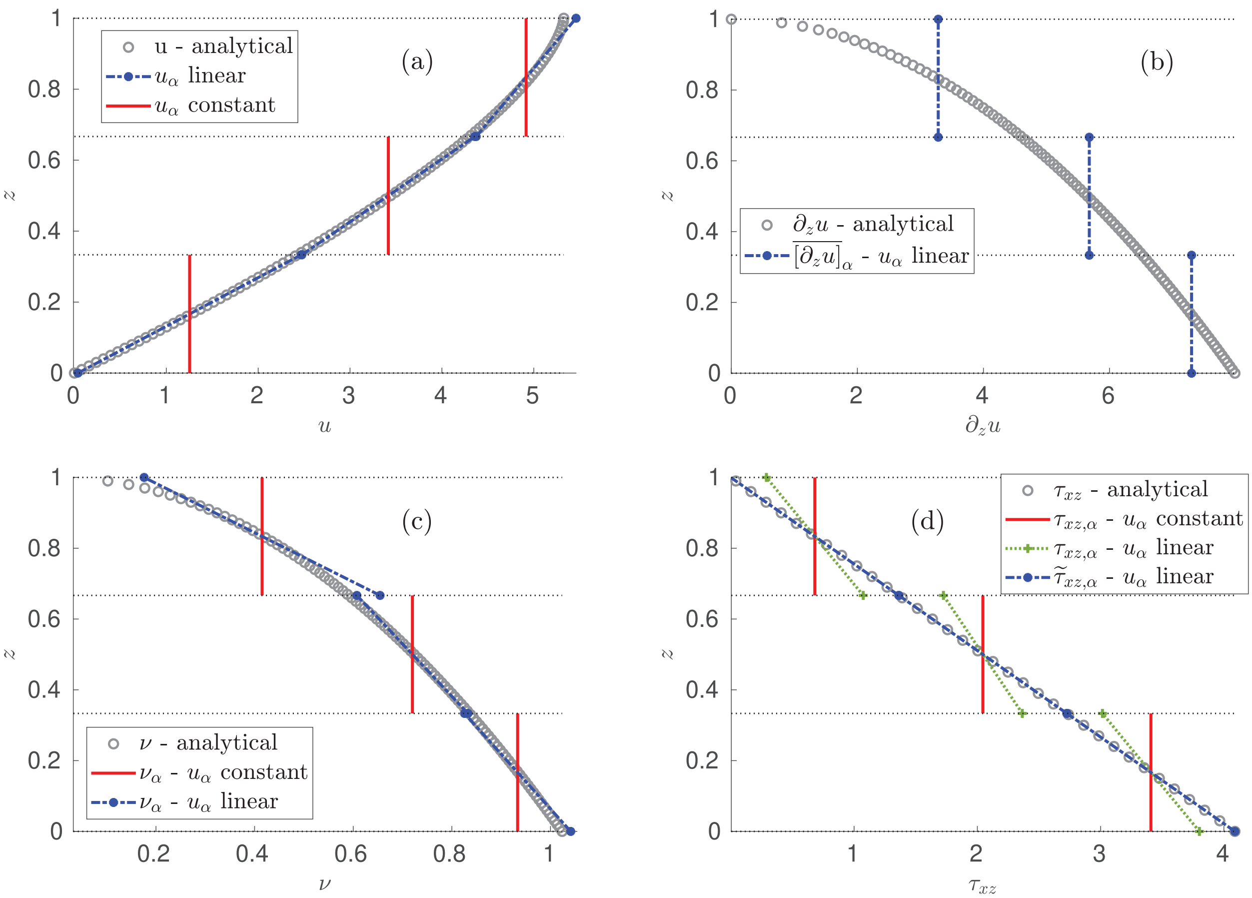

FIGURE 3. Comparison between the analytical vertical profiles (grey circles) and layerwise approximations with constant (solid red lines) and linear (dot-dashed blue and dotted green lines) horizontal velocity, and 3 layers: (a) Horizontal velocity (u); (b) vertical derivative ($\partial_z u$); (c) viscosity coefficient (ν); (d) stress tensor component τ_{xz}, where $\widetilde{\tau}_{xz,\alpha}$ denotes the second-order correction of $\tau_{xz,\alpha}$ (33).

and the following profiles are obtained (see [28, 37])

$$\begin{cases} u_{\text{an}}(z) = \frac{2}{3} I_\theta \left(H^{3/2} - (H-z)^{3/2} \right), \\ \partial_z u_{\text{an}}(z) = I_\theta \sqrt{H-z}, \\ \tau_{xz,\text{an}}(z) = g \sin\theta (H-z), \end{cases}$$

with

$$I_\theta = \frac{I_0}{d_s} \left(\frac{\tan\theta - \mu_s}{\mu_2 - \tan\theta} \right) \sqrt{\varphi_s g \cos\theta},$$

being $d_s, \varphi_s, I_0, \mu_s, \mu_2$ constant parameters depending on the granular material.

Let us consider a test as in [37], where a flow with height $H = 1$ is considered. The slope is taken as $\theta = 0.43 \approx 23.3°$. The rheological parameters are $d_s = 0.04$, $\varphi_s = 0.62$, $Io = 0.279$, $\mu_s = 0.38$ and $\mu_2 = 0.62$.

Figure 3 shows the agreement between the layerwise linear approach and the analytical solutions for this case, also with 3 vertical layers. We also show here the approximation of the viscosity coefficient. We observe that using a layerwise constant coefficient leads to an important loss of accuracy in this case. Similar conclusions as in the Newtonian case are obtained. Let us remark again the importance of the correction in $\widetilde{\tau}_{xz,\alpha}$ with respect to $\tau_{xz,\alpha}$, making possible to perfectly approximate the analytical solution $\tau_{xz,\mathrm{an}}$. Note that $\tau_{xz,\alpha}$ with layerwise linear velocity is not able to properly approximate the slope of $\tau_{xz,\mathrm{an}}$, although it is no more layerwise constant as in the Newtonian case. Moreover, it is not solved by increasing the number of vertical layers.

6.3. Herschel–Bulkley viscoplastic fluids

Concerning Herschel–Bulkley fluids, the regularised viscosity coefficient is given by

$$\nu_{\mathrm{an}}(z) = \frac{\tau_y + K|\partial_z u|^n}{\sqrt{|\partial_z u|^2/4 + \delta^2}},$$

with τ_y, K and n constant rheological parameters. These flows are characterised by a top (pseudo-)plug layer of material, where the stress tensor is not defined. Actually, we only know that $|\partial_z u| = 0$ and $|\boldsymbol{\tau}| < \tau_y$ there. Then, the flow can be split into a lower sheared layer with height h_c defined by

$$h_c = H - \frac{\tau_y}{g\sin\theta},$$

and the (pseudo-)plug top layer, with thickness $H - h_c$. Then, from (49), the analytical solution reads (see [10,12]) for $z < h_c$

$$\begin{cases} u_{\mathrm{an}}(z) = u_{\mathrm{plug}}\left(1 - \left(1 - \frac{z}{h_c}\right)^{(n+1)/n}\right), \\ \partial_z u_{\mathrm{an}}(z) = \left(\frac{g\sin\theta}{K}\right)^{1/n}(h_c - z)^{1/n}, \\ \tau_{xz,\mathrm{an}}(z) = \rho g \sin\theta(H - z), \end{cases}$$

and we have

$$u_{\mathrm{an}}(z) = u_{\mathrm{plug}}, \qquad \partial_z u_{\mathrm{an}} = 0, \quad \text{and} \quad |\tau_{xz,\mathrm{an}}(z)| \le \tau_y,$$

for $z \ge h_c$ with

$$u_{\mathrm{plug}} = \frac{n}{(n+1)}\left(\frac{g\sin\theta}{K}\right)^{1/n} h_c^{(n+1)/n}.$$

In this case we consider a test as in [20], where a material with height $H = 0.05$ and a slope with angle $\theta = 20°$ are taken. The rheological parameters are $\tau_y = 0.033$, $K = 0.026$ and $n = 0.33$. Figure 4 shows the results, in this case for 4 vertical layers.

Let us mention that this case involves some important difficulties related to the existence of the pseudo-plug layer. In particular, non-physical values for the viscosity ($\nu_\alpha(z) < 0$) are obtained due to the strong transition occurring at $z = h_c$ (see the inner figure in Fig. 4c). Therefore, some limiters are needed in order not to have this kind of meaningless behaviour. Concretely, we make the following corrections when needed (in practice, it is only necessary in the layer containing the level $z = h_c$ and/or the layer below):

(i) The signs of $\overline{[\partial_z u]}_\alpha$ and λ_α must be the same. If it does not hold, the vertical derivative is redefined as $\overline{[\partial_z u]}_\alpha = \lambda_\alpha$.
(ii) Negative values of the viscosity coefficient are physically meaningless, so if $\nu_\alpha(z_0) < 0$ for $z_{\alpha-1/2} \le z_0 \le z_{\alpha+1/2}$, the viscosity is taken as constant with the averaged value $\nu_\alpha(z) = \nu_\alpha^0 > 0$.
(iii) The signs of $\tau_{xz,\alpha}(z)$ and $\overline{[\partial_z u]}_\alpha$ must be equals. If this condition is violated somewhere, we again remove the slope of $\tau_{xz,\alpha}(z)$ in that layer by setting $\tau_{xz,\alpha}(z) = \overline{\tau}_{xz,\alpha} > 0$ constant. The same applies for $\widetilde{\tau}_{xz,\alpha}$.

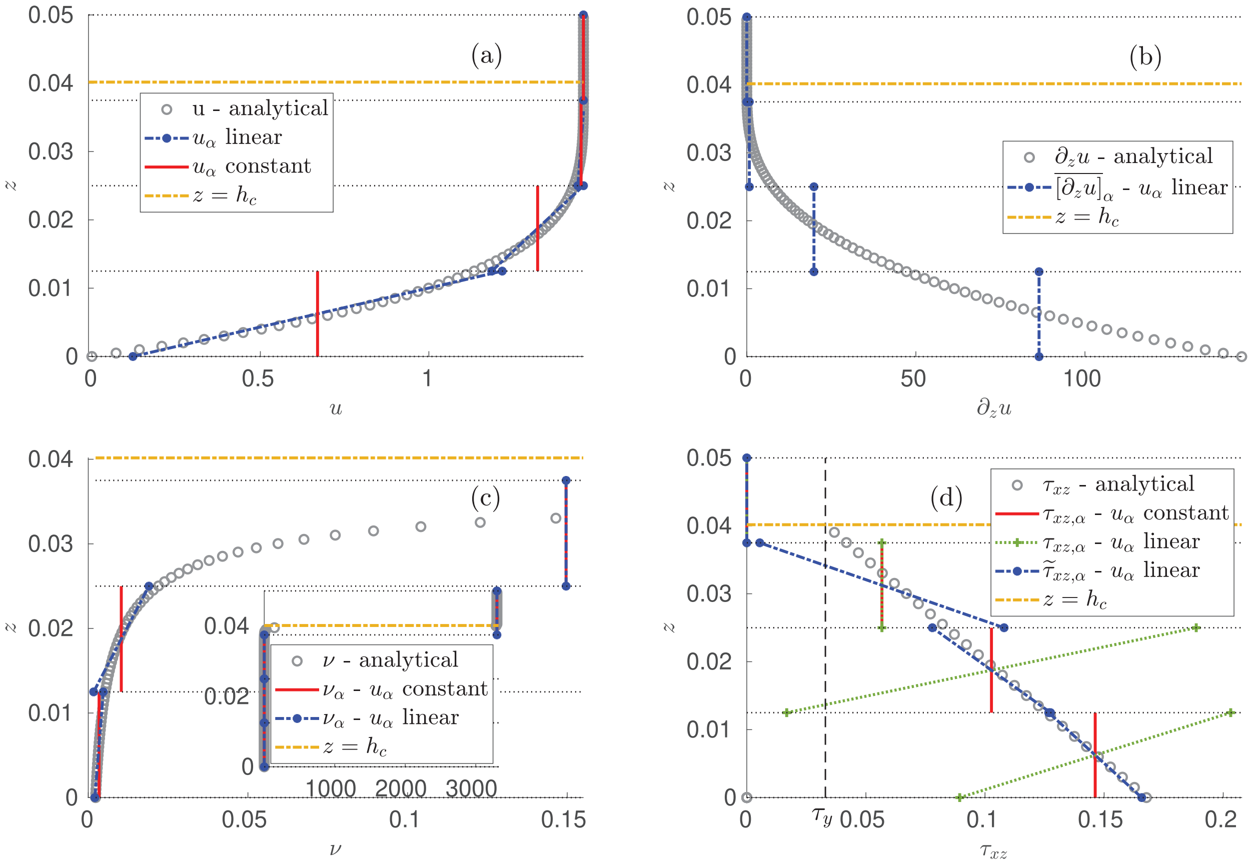

FIGURE 4. Comparison between the analytical vertical profiles (grey circles) and layerwise approximations with constant (solid red lines) and linear (dot-dashed blue and dotted green lines) horizontal velocity, and 4 layers: (a) Horizontal velocity (u); (b) vertical derivative ($\partial_z u$); (c) viscosity coefficient (ν); (d) stress tensor component τ_{xz}, where $\widetilde{\tau}_{xz,\alpha}$ denotes the second-order correction of $\tau_{xz,\alpha}$ (33). Dot dashed cyan lines represent the thickness of the sheared layer $z = h_c$.

Note that the idea behind conditions (ii) and (iii) is to reduce the complexity by going to a constant profile, but it is different from a layerwise constant approach, since the information related to the slope λ_α is taken into account for ν_α^0 and $\overline{[\partial_z u]}_\alpha$, and therefore for $\overline{\tau}_{xz,\alpha}, \widetilde{\tau}_{xz,\alpha}$.

Figures 4a and 4b show the profiles of $u(z)$ and $\partial_z u(z)$. We can distinguish the sheared and pseudo-plug layers connected at the level $z = h_c$. In Figure 4c the viscosity profile is depicted. Note that this is a zoom of the sheared layer, and the inner figure there shows the global profile. One can see there the effect of the regularisation method, which makes the viscosity quickly increase in the top pseudo-plug layer. It is due to the fact that we have $\partial_z u = 0$ in this layer. This behaviour of the viscosity has a great impact on the shear stress τ_{xz}, as we see in Figure 4d. As in previous cases, we see how the corrected shear stress $\widetilde{\tau}_{xz,\alpha}$ is able to approximate the linear profile of τ_{xz} much better than $\tau_{xz,\alpha}$, although the accuracy seems to be lost close to $z = h_c$.

In Figure 5 we see the same comparison for the shear stress when increasing the number of vertical layers. Concretely, we use 8 and 16 layers. We see that the profile of τ_{xz} is properly approximated by $\widetilde{\tau}_{xz,\alpha}$ (and not by $\tau_{xz,\alpha}$) in the sheared layer when increasing the vertical resolution, whereas the accuracy is only lost close

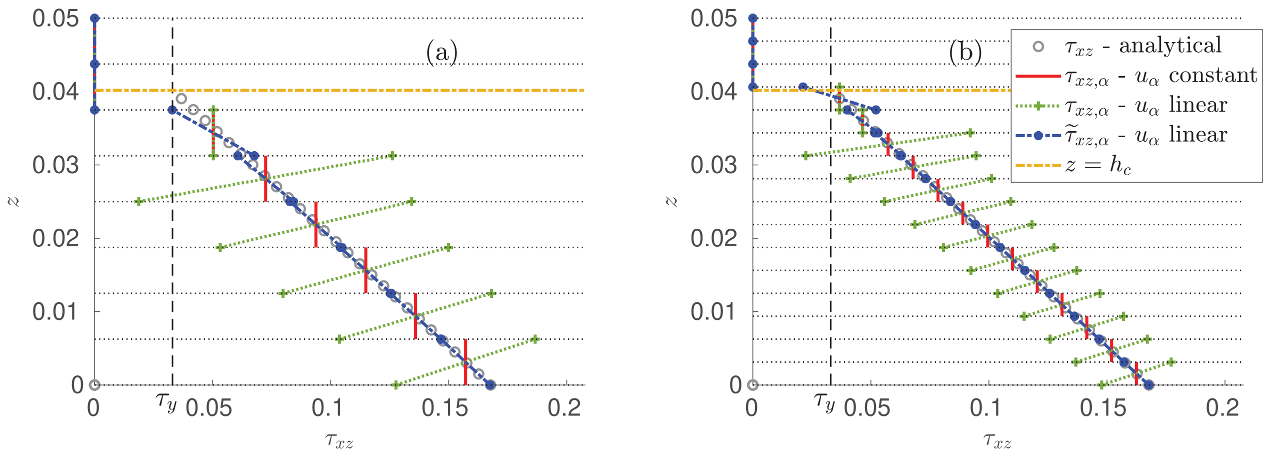

FIGURE 5. Comparison between the analytical vertical profiles (grey circles) and layerwise approximations with constant (solid red lines) and linear (dot-dashed blue and dotted green lines) horizontal velocity for the stress tensor component τ_{xz}, where $\widetilde{\tau}_{xz,\alpha}$ denotes the second-order correction of $\tau_{xz,\alpha}$ (33). (a) 8 layers; (b) 16 layers.

to the interface at $z = h_c$. Here we see again the key role of the second-order correction in $\widetilde{\tau}_{xz,\alpha}$, since $\tau_{xz,\alpha}$ cannot reproduce the linear profile despite increasing the number of layers. It means that this lack of accuracy is associated with the definition of $\tau_{xz,\alpha}$ and not with the vertical resolution.

7. CONCLUSIONS

Several layer-averaged models with layerwise linear horizontal velocity and non-hydrostatic pressure for the Navier–Stokes system are derived in this paper. We focus on an appropriate definition of the terms that come from viscous contributions for a general stress tensor. In particular, we give detailed definitions of all components of the stress tensor when it is proportional to the strain rate tensor, which is very common in fluid dynamics. In that case, the approximations of the derivatives of the velocity (20) are inspired by the theory of distributions, in order to account for the possible discontinuities of the velocity at the internal interfaces $\mathcal{L}_{\alpha+1/2}$, for $\alpha = 1, \ldots, L-1$. Following the usual layer-averaging procedure, the LIN-NH$_2$-STRESS model is derived, as well as its simplified version LIN-NH$_1$-STRESS. Both models have been also written in a compact form, where we observe two terms related to the stress tensor approximation. The first one is a diffusion term, which includes second order derivatives of the velocity components in the case of a viscous stress tensor, and the second one corresponds to the momentum transference terms at the interfaces. LIN-NH$_k$-STRESS models are the extension of those models introduced in [15], denoted by LIN-NH$_k$ with $k = 1, 2$, to the Navier–Stokes case. These models satisfy a dissipative energy balance, where the right-hand side is written in integral form. An important remark is the fact that all terms in these models are approximated to second-order accuracy, except for $\tau_{xz,\alpha}$. Concretely, it is a first-order approximation, due to the fact that $\partial_z u$ is layerwise constant. However, we also propose a correction allowing us to obtain the second-order accuracy. This corrected model satisfies a dissipative energy balance up to second order.

These and other models are also obtained from an asymptotic analysis of the Navier–Stokes system, for different orders of magnitude of the shallowness parameter (ε). An interesting model, denoted here LIN-H-STRESS, is obtained when removing the vertical acceleration but keeping viscous terms in the pressure. The resulting model in this case is closely related to the one introduced in [8], where both horizontal and vertical velocities are layerwise constant. Actually, it is a generalisation of that model to the layerwise linear velocity and viscosity. We have also introduced a hydrostatic version of these models with the vertical diffusion term.

In conclusion, the models presented in this paper are generalisations of plenty of previous models in the literature, in the framework of depth(and layer)-averaged models for geophysical flows, as explained in Section 5.4. Let us mention that all these models satisfy dissipative energy balances. We also remark that the presented non-hydrostatic models with layerwise linear horizontal velocity have the same excellent dispersion properties (dispersion relation, velocity group and linear shoaling) as LIN-NH$_k$ models (see [15]).

We have also shown how the proposed layerwise linear approach is effective for some geophysical flows, including complex viscoplastic fluids, in the uniform configuration, where it is possible to get analytical solutions. This approach allows us to notably improve the approximation of the vertical profile of velocity with respect to the layerwise constant approach. Accurate approximations are reached by using just a few layers with this novel approach. Interestingly, we have also seen that the second-order correction for the shear stress τ_{xz} is essential to be able to recover the linear profile of its analytical solution. Actually, it is not possible without this correction, even for simple Newtonian fluids.

The models presented in this work have to be analyzed before discretizing them, for instance following the results introduced in [14] for a multilayer model with a simpler definition of the viscous terms. In the future, it would be also interesting to develop efficient numerical schemes, following the strategies introduced in [17], as well as the extension of this framework to approximate 3D Navier–Stokes systems and more general stress tensor definitions, such as the case of turbulent models.

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