On the geometry of stationary Galilean spacetimes

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Abstract

In this work we introduce a new family of non-relativistic spacetimes: standard stationary Galilean spacetimes, which constitute the local geometric model of stationary Galilean spacetimes. We also study the geodesic completeness of stationary Galilean spacetimes as well as the the geometric conditions for these spacetimes that guarantee the existence of a global splitting as a standard stationary Galilean spacetime.

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1 Introduction

The geometrization of Newton's theory of gravity began with the works of Cartan [6, 7] and Friedrichs [13]. Since then, many authors have studied these spacetime models by their own interest as well as compared to the ones in Einstein's General Relativity (see, for instance, [3], [9], [10], [15], [16], [21] and [25]).

Newton-Cartan theory shows that some features of General Relativity are also shared by these models. Indeed, classical Newtonian gravitation is formulated as a covariant theory where gravity also appears as a consequence of the curvature of a connection in the spacetime (which does not come from a semi-Riemannian metric) and the spacetime structure is dynamical, in the sense that it is not a fixed scenario in which physics unfolds but participates in this unfolding [18]. Moreover, this formulation enables us to establish in an accurate and intrinsic way the limit relation between the Newtonian theory of gravitation and General Relativity [17]. Furthermore, Newton-Cartan theory of gravity provides a valuable setting for exploring some fundamental problems of the elusive full quantum theory of gravity [8]. In fact, possible applications of the models described in this article might include the comparison between two similar stationary Newtonian spacetimes, which appears in simple models of quantum collapse [20].

In order to obtain physically relevant spacetime models it is usual to assume the existence of a symmetry given by the existence of a vector field whose flow preserves the structure of the spacetime [11]. In General Relativity, this symmetry can be ensured by existence of a timelike Killing vector field in a Lorentzian spacetime, since the stages of the local flow of this Killing vector field are isometries. Lorentzian spacetimes admitting a timelike Killing vector field are called stationary spacetimes [22]. Furthermore, if the timelike Killing vector field is also irrotational the spacetime is called static [19]. Stationary spacetimes can be locally written as a product manifold with an appropriate non-orthogonal semi-Riemannian metric. Moreover, under natural causality assumptions this local splitting can be global [14], obtaining a standard stationary spacetime.

In the non-relativistic setting, the natural symmetry is provided by timelike vector fields whose flows preserve the Galilean structure of the spacetime, which are called timelike Galilean vector fields. Galilean spacetimes admitting a timelike Galilean vector field are called stationary (see Definition 1). Moreover, if this vector field is also irrotational the spacetime is called static.

Stationary Galilean spacetimes model a non-relativistic universe where the associated field of observers to the timelike Galilean vector field measure that the spatial metric and inertia do not change with time. Moreover, in this article we introduce the concept of standard stationary spacetime, which represents the prototypical stationary spacetime (Definition 7). Indeed, we prove in Proposition 12 that every stationary Galilean spacetime locally has a standard stationary structure. We should point out that the decomposition of a stationary spacetime as a standard one may not be unique (some uniqueness results in the analogous relativistic setup may be found in [1], [2] and [23]). Nevertheless, along this article, when speaking of a standard stationary spacetime, all the elements of the decomposition will be assumed to be prescribed.

The aim of this article is to study the properties of stationary Galilean spacetimes as well as introduce a new family of geometric Galilean models: standard stationary Galilean spacetimes. In order to better understand the geometry of these models we devote Section 2 to summarize the basic setup of Newton-Cartan theory and in Section 3 we obtain several results for stationary and static Galilean spacetimes in general. Section 4 is devoted to introduce the definition of standard stationary and standard static Galilean spacetime as well as to prove in Proposition 12 how every stationary Galilean spacetime locally presents a standard stationary structure. Furthermore, in Section 5 we provide some geometric conditions that guarantee that a stationary Galilean spacetime admits a global splitting as a standard stationary Galilean spacetime. To conclude, we study the geodesic completeness of stationary Galilean spacetimes (Theorem 17) and particularize our results for the standard case in Section 6.

2 Preliminaries

A *Leibnizian* spacetime is the triad, (M, Ω, g), where M^{n+1} is a smooth connected manifold of arbitrary dimension $n+1 \geq 2$ endowed with the Leibnizian structure (Ω, g), being $\Omega \in \Lambda^1(M)$ a nowhere null differential 1-form ($\Omega_p \neq 0$, $\forall p \in M$) and g a positive definite metric on the kernel of Ω. Specifically, denoting by $\mathrm{An}(\Omega) = \{v \in TM\,,\, \Omega(v) = 0\}$ the smooth n-distribution induced on M by Ω and by $\Gamma(TM)$ the set of smooth vector fields on M, we may construct the subset $\Gamma(\mathrm{An}(\Omega)) = \{V \in \Gamma(TM) \,/\, V_q \in \mathrm{An}(\Omega),\, \forall p \in M\}$. Hence, the map

$$g : \Gamma(\mathrm{An}(\Omega)) \times \Gamma(\mathrm{An}(\Omega)) \longrightarrow C^{\infty}(M),\ (V, W) \mapsto g(V, W),$$

is smooth, bilinear, symmetric and positive definite (see [3] and [4] for details).

A point $p \in M$ is called an *event*. The Euclidean vector space $(\mathrm{An}(\Omega_p)\,, g_p)$ is called the *absolute space* at $p \in M$, and the linear form Ω_p is the *absolute clock* at p. A tangent vector $v \in T_pM$ is *spacelike* if $\Omega_p(v) = 0$ and, otherwise, *timelike*. In addition, if $\Omega_p(v) > 0$ (resp., $\Omega_p(v) < 0$), v points to the *future* (resp., to the *past*).

An *observer* in a Leibnizian spacetime is a smooth curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ whose velocity γ' is a unitary future pointing timelike vector field, that is, $\Omega(\gamma'(s)) = 1$ for all $s \in I$. The parameter s is called the *proper time* of the observer γ. A vector field $Z \in \Gamma(TM)$ with $\Omega(Z) = 1$ is called a *field of observers*, i.e., its integral curves are observers.

When the smooth distribution $\mathrm{An}(\Omega)$ is integrable (equivalently, if the absolute clock Ω satisfies $\Omega \wedge d\Omega = 0$), the Leibnizian spacetime (M, Ω, g) is said to be *locally synchronizable* and, from the Frobenius Theorem (see [26]), it can be foliated by a family of hypersurfaces tangent to the absolute space $\{\mathcal{F}_\lambda\}$. In this case, it is well-known that each $p \in M$ has a neighborhood where $\Omega = \beta\ dT$, for certain smooth functions $\beta > 0$, T, and the hypersurfaces $\{T = \text{constant}\}$ locally coincide with a leaf of the foliation $\mathcal{F}$. Therefore, any observer can rescale its proper time to be synchronized with the "compromise time" T. In the more restrictive case where $d\Omega = 0$ the Leibnizian spacetime is called *proper time locally synchronizable* and, locally, $\Omega = dT$. In this case observers are synchronized directly by its proper time (up to a constant). When $\Omega = dT$ for some function $T \in C^{\infty}(M)$, any observer may be assumed to be parametrized by the *absolute time function* T. Notice that the notion of (local and local proper time) synchronizability is intrinsic to the Leibnizian structure, applicable for every observer, in contrast to the relativistic setting, where the analogous concepts only have meaning for fields of observers.

According to [4], a vector field K is called *Leibnizian* if the stages Φ_s of its local flows are *Leibnizian diffeomorphisms*, that is, they preserve the absolute clock and space, i.e.,

$$\Phi_s^*\Omega = \Omega, \qquad \text{and} \qquad \Phi_s^* g = g.$$

Both conditions are equivalent to the following ones,

(i) $\Omega([K, X]) = K(\Omega(X)), \quad \forall X \in \Gamma(TM)$.

(ii) $K(g(V, W)) = g([K, V], W) + g(V, [K, W]), \quad \forall V, W \in \Gamma(\mathrm{An}(\Omega))$.

Notice that $[K, V]$, $[K, W] \in \Gamma\big(\mathrm{An}(\Omega)\big)$ by (i), so (ii) is always well defined.

On the other hand, the inertia principle must be codified through a connection on the spacetime. However, since a Leibnizian structure does not have a canonical affine connection associated, it is required to introduce a compatible connection with the absolute clock Ω and the space metric g, i.e., a connection ∇ such that

(a) $\nabla\Omega = 0$ (equivalently, $\Omega(\nabla_X Y) = X(\Omega(Y))$ for any $X, Y \in \Gamma(TM)$).

(b) $\nabla g = 0$ (i.e., $Z(g(V,W)) = g(\nabla_Z V, W) + g(\nabla_Z W, V)$ for any $Z \in \Gamma(TM)$ and V, W spacelike vector fields).

Such a connection is called *Galilean* and its restriction to the spacelike leaves of the foliation coincides with the Levi-Civita connection associated to g. A *Galilean spacetime* (M, Ω, g, ∇) is a Leibnizian spacetime endowed with a Galilean connection ∇. In addition, ∇ is said to be *symmetric* if its torsion vanishes identically ($\mathrm{Tor}_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \equiv 0$). Notice that thanks to [4, Lemma 13] and [4, Cor. 28], the existence of a symmetric Galilean connection for a Leibnizian structure is equivalent to the proper time local synchronizability of the latter. Moreover, making use of Poincaré's lemma, it is clear that if the spacetime is simply-connected then there exists an absolute time function. On the other hand, from a physical standpoint, a symmetric connection is desirable since it is completely determined by its geodesic trajectories, i.e., by the free falling observers in M.

Given two Galilean spacetimes (M, Ω, g, ∇) and $(M', \Omega', g', \nabla')$, a diffeomorphism $F : M \longrightarrow M'$ is said to be *Galilean* if $F^*\Omega' = \Omega$, $F^*g' = g$ and $F^*\nabla' = \nabla$, i.e., $\nabla'_{dF(X)} dF(Y) = \nabla_X Y$.

For any field of observers Z on a Galilean spacetime (M, Ω, g, ∇), the *gravitational field* induced by ∇ in Z is given by the spacelike vector field $\mathcal{G} = \nabla_Z Z$ and the *vorticity* or *Coriolis field* of Z is the $2-$form $\omega(Z) = \frac{1}{2}\mathrm{Rot}(Z)$, defined as

$$\omega(Z)(V,W) = \frac{1}{2}\Big(g(\nabla_V Z, W) - g(\nabla_W Z, V)\Big) \qquad \forall V, W \in \Gamma(\mathrm{An}(\Omega)).$$

Note that $\omega(Z)$ is well defined because $\Omega(Z)$ is constant and that Z is called irrotational when ω vanishes.

The importance of the gravitational field and the vorticity of a field of observers is due to the fact that they determine a unique symmetric Galilean geometry of proper time locally synchronizable spacetimes [4, Cor. 28]. Furthermore, that symmetric Galilean connection admits a formula 'à la Koszul' for the field of observers Z whose expression is

$$\nabla_X Y = P^Z(\nabla_X Y) + X(\Omega(Y))Z, \qquad \forall X, Y \in \Gamma(TM), \tag{1}$$

where $P^Z(X) = X - \Omega(X)Z$ is the natural spacelike projection for Z and, for each $V \in \Gamma(\mathrm{An}(\Omega))$,

$$\begin{aligned} 2g(P^Z(\nabla_X Y), V) \;=\;& X(g(P^Z(Y), V)) + Y(g(P^Z(X), V)) - V(g(P^Z(X), P^Z(Y)) \\ &+ 2\,\Omega(X)\Omega(Y)g(\mathcal{G}, V) + 2\,\Omega(X)\omega(Z)(P^Z(Y), V) + 2\,\Omega(Y)\omega(Z)(P^Z(X), V) \\ &+ \Omega(X)\left(g([Z, P^Z(Y)], V) - g([Z, V], P^Z(Y))\right) \\ &- \Omega(Y)\left(g([Z, P^Z(X)], V) + g([Z, V], P^Z(X))\right) \\ &+ g([P^Z(X), P^Z(Y)], V) - g([P^Z(Y), V], P^Z(X)) - g([P^Z(X), V], P^Z(Y)). \end{aligned}$$

Moreover, a Leibnizian vector field K in a Galilean spacetime (M, Ω, g, ∇) is called *Galilean* if it is affine for ∇, that is, $L_K \nabla = 0$, where L denotes the Lie derivative. This condition may be characterized as follows:

$$[K, \nabla_Y X] = \nabla_{[K,Y]} X + \nabla_Y [K, X], \qquad \forall X, Y \in \Gamma(TM). \tag{2}$$

Finally, a Galilean spacetime is said to be *Newtonian* if it admits an irrotational Galilean field of observers and the (symmetric) connection ∇ restricted to the spacelike vectors is flat. Newtonian spacetimes have traditionally represented the classical (non-relativistic) geometric models of gravity.

3 Stationary and static Galilean spacetimes

We present now the classical version of the well known relativistic stationary and static spacetimes.

Definition 1 *A Galilean spacetime (M, Ω, g, ∇) is said to be* stationary *if admits a future pointing Galilean vector field K, i.e., a Leibnizian vector field such that*

$$L_K \nabla = 0.$$

Such a vector field K is called a stationary vector field. *In addition, if the associated field of observers $Z = K/\Omega(K)$ is irrotational, the Galilean spacetime (M, Ω, g, ∇) will be called* static *with respect to K.*

The adjective "stationary" on these kinds of spacetimes means that the associated field of observers to the Galilean vector field K locally measure that the spatial metric g and the inertia ∇ do not change with the local absolute time as we can see in the following proposition. In addition, the static case happens when these Galilean observers do not rotate.

Proposition 2 *Let (M, Ω, g, ∇) be a locally synchronizable stationary Galilean spacetime with respect to the timelike vector field K, and let $Z = K/\Omega(K)$ be its associated vector field of observers. Then, for each $p \in M$ there exists a chart $(U; (t, x_1, \cdots, x_n))$, with $p \in U$, such that the local vector fields ∂_{x_i}, $i = 1, ..., n$ are spacelike, and*

$$K = \partial_t, \quad \Omega = \Omega(K) dt, \quad Z(g_{ij}) = 0, \quad Z(\Gamma^k_{\mu\nu}) = 0, \tag{3}$$

for $1 \leq i, j \leq n$, $0 \leq \mu, \nu, k \leq n$, where $g_{ij} = g(\partial_i, \partial_j)$ and $\Gamma^k_{\mu\nu}$ are the Christoffel symbols of ∇ in $(U; (t, x_1, \cdots, x_n))$.

Proof. Fix a local chart $(W; y_0, y_1, \cdots, y_n)$ such that $K|_W = \partial_{y_0}$ and define

$$V_i = \partial_{y_i} - \Omega(\partial_{y_i}) Z \in \Gamma(\mathrm{An}(\Omega_{|W})), \quad 1 \leq i \leq n. \tag{4}$$

Since the vector fields V_i, $i = 1, ..., n$ are spacelike, we observe that

$$d\Omega(V_i, V_j) = -\Omega([V_i, V_j]), \quad \forall i, j \in \{1, \ldots, n\}.$$

On other hand, as the smooth distribution $\mathrm{An}(\Omega)$ is involutive it follows that $\Omega([V_i, V_j]) = 0$ and from (4), it is easy to see that $[V_i, V_j] = 0$. Moreover, thanks to the Leibnizian character of K we have $\Omega([K,V_i])=0$, $i=1,...,n$, but again from (4) it is not difficult to see that $[K,V_i]=0$, $i=1,...,n$. Thus, making use of [24, Th 14, Chap. 5], we can ensure the existence of a chart $(U;(t,x_1,\cdots,x_n))$, $p\in U\subseteq W$, such that

$$K=\partial_t,\ \partial_i:=\partial_{x_i}=V_i,\quad 1\le i\le n.$$

Consequently, $\Omega=\Omega(K)dt$. It may be also noticed $\Omega(K)$ is independent of t from condition (i) for K.

Finally, from condition (ii) follows $K(g_{ij})=0$, $\forall i,j\in\{1,\dots,n\}$. Similarly, inserting $X=\partial_\mu$ and $Y=\partial_\nu$ in (2), we conclude that $K(\Gamma^k_{\mu\nu})=0$, $0\le\mu,\nu,k\le n$.

□

Remark 3 Observe that the Galilean character of the Leibnizian vector field K is only needed to prove the last property for the Christoffel symbols.

As a direct consequence of Proposition 2 we obtain,

Corollary 4 *A locally synchronizable Galilean spacetime which is stationary with respect to K is locally proper time synchronizable if and only if $\Omega(K)$ is a constant function.*

Remark 5 Let (M,Ω,g,∇) be a Galilean spacetime and let $K\not\equiv 0$ be a Galilean vector field. If we assume $d\Omega=0$, the Leibnizian character of K ensures that the function $\Omega(K)=c$ is constant. Therefore, if $c\neq 0$ then K is stationary and up to a sign we can normalize $c=1$. Moreover, if there are no more independent stationary vector fields, the unique Galilean vector field $\tilde{K}$ satisfying $\Omega(\tilde{K})\equiv 0$ is $\tilde{K}\equiv 0$.

Let (M,Ω,g,∇) be a locally synchronizable stationary Galilean spacetime with respect to a Galilean vector field K, and let $Z=K/\Omega(K)$ be its associated vector field of observers. If we consider the gravitational field $\mathcal{G}=\nabla_Z Z$, it is natural to investigate the change of this vector field along the integral curves of Z. Taking into account the Leibnizian character of K we have that $\mathcal{G}=\frac{1}{\Omega(K)^2}\nabla_K K$. Since K is a Galilean vector field, it is not difficult to see that

$$L_Z\mathcal{G}=-\frac{1}{\Omega(K)^4}d\Omega(K,\nabla_K K)K. \tag{5}$$

In particular, if Ω is closed we obtain $L_Z\mathcal{G}=0$.

Analogously, consider the Coriolis field of Z, $\omega(Z)$. The Lie derivative $L_Z\omega$ will only make sense when $\Omega([Z,V])=0$ for any spacelike vector field V. Consequently, from Proposition 2, $\Omega([Z,V])=0$ for any spacelike vector field V if and only if $d\Omega=0$. In particular, if the stationary vector field is a field of observers we obtain the next result.

Proposition 6 *Let (M,Ω,g,∇) be a locally synchronizable stationary Galilean spacetime with respect to a Galilean vector field of observers Z. Then,*

$$L_Z\mathcal{G}=0\qquad\text{and}\qquad L_Z\omega=0, \tag{6}$$

where $\mathcal{G}=\nabla_Z Z$ and $\omega=\frac{1}{2}\mathrm{Rot}(Z)$ are the gravitational field and the vorticity of Z, respectively.

Proof. Substituting $K = X = Y = Z$ in (2) we deduce that $[Z, \nabla_Z Z] = 0$, which is the first equality in (6).

For the second identity, taking into account Proposition 2, the absolute clock an Z may be locally expressed as $\Omega = dt$ and $Z = \partial_t$. Using the previous adapted chart we obtain

$$2L_Z\omega(\partial_i, \partial_j) = 2\partial_t(\omega(\partial_i, \partial_j)) =$$
$$\partial_t\big(g(\nabla_{\partial_i}\partial_t, \partial_j) - g(\nabla_{\partial_j}\partial_t, \partial_i)\big) = g\Big([\partial_t, \nabla_{\partial_i}\partial_t], \partial_j\Big) + g\Big([\partial_t, \nabla_{\partial_j}\partial_t], \partial_i\Big) = 0.$$

□

4 Standard stationary Galilean spacetimes

In this section we will introduce two families of geometric models, which represent the stationary and static prototypical Galilean spacetimes.

Definition 7 *Let $I \subseteq \mathbb{R}$ be a real interval, (S, h) an n-dimensional connected Riemannian manifold and $\pi_I : I \times S \longrightarrow I$ and $\pi_S : I \times S \longrightarrow S$ the canonical projections. A Galilean spacetime (M, Ω, g, ∇) is called a* standard stationary Galilean spacetime *if*

1. *$M = I \times S$,*
2. *$\Omega = (\alpha \circ \pi_S)\, d\pi_I$, where α is a smooth positive function defined on S,*
3. *g is the restriction to the bundle $\mathrm{An}(\Omega)$ of the (degenerate) metric on M,*

$$\overline{g} = \pi_S^* h, \tag{7}$$

4. *∇ is a Galilean connection on (M, Ω, g) satisfying*

$$[\partial_t, \nabla_Y X] = \nabla_{[\partial_t, Y]}X + \nabla_Y[\partial_t, X], \qquad \forall X, Y \in \Gamma(TM),$$

where ∂_t is the global coordinate vector field associated to $t := \pi_I$. Moreover, if ∂_t/α is also irrotational the spacetime will be called a standard static Galilean spacetime.

In the following, with an abuse of notation we will write αdt to denote $(\alpha \circ \pi_S)d\pi_I$.

Remark 8 In a standard stationary or static Galilean spacetime a (future pointing) timelike Galilean vector field corresponds with $K = \partial_t$, which is clearly Leibnizian. Note that our definition of standard stationary Galilean spacetime implies local synchronizability. Furthermore, from Definition 7 we obtain $d\Omega = d\alpha \wedge dt$ and trivially:

Proposition 9 *Let $(I \times S, \Omega = \alpha dt, g, \nabla)$ be a standard stationary Galilean spacetime. Then, the connection ∇ is symmetric if and only if the function $\alpha \in C^\infty(S)$ is constant.*

Remark 10 Therefore, in a standard stationary Galilean spacetime if α is not constant then the connection cannot be symmetric. This fact is an important difference with the relativistic case and enables us to find examples of stationary Galilean spacetimes with non constant $\Omega(K)$. However, the converse is not true, which can be easily seen from the fact that the usual Leibnizian structure of $\mathbb{R}^{n+1}$ admits non-symmetric Galilean connections [4, Thm. 27].

Example 11 Let us consider $I = \mathbb{R}$ and $S = \mathbb{R}^3$ endowed with the usual Euclidean metric $g_{\mathbb{R}^3}$, as well as a positive function with compact support $\rho \in C_c^\infty(I \times S)$ such that $\partial_t(\rho) = 0$. Let us consider the standard Cartesian coordinates in $\mathbb{R} \times \mathbb{R}^3$ and denote them by (t, x_1, x_2, x_3) and let G be the Universal Gravitational constant. The Galilean spacetime $(I \times S, dt, \pi_{\mathbb{R}^3}^* g_{\mathbb{R}^3}, \nabla)$, where ∇ is the unique symmetric connection verifying

$$\mathcal{G} = \nabla_{\partial_t}\partial_t = G \sum_{i=1}^{3} \left(\int_{\mathbb{R}^3} \frac{\rho(x_1', x_2', x_3')\,(x_i - x_i')}{\left(\sum_{j=1}^{3}(x_j - x_j')^2\right)^{3/2}}\, dx_1' dx_2' dx_3' \right) \partial_i$$

and

$$\mathrm{Rot}(\partial_t) = 0$$

is a standard static Galilean spacetime, as it may be directly checked. It physically represents a Newtonian spacetime where there is a mass distribution with density ρ, which is static for the field of observers ∂_t.

The next proposition justifies why standard stationary Galilean spacetimes are the prototype of stationary Galilean spacetimes.

Proposition 12 *Every locally synchronizable stationary Galilean spacetime (M, Ω, g, ∇) is locally, a standard stationary Galilean spacetime, i.e., for each $p \in M$ there exist a neighborhood U and a Galilean diffeomorphism $F : N \longrightarrow U$, where N is a standard stationary spacetime.*

Proof. Let F_s be the local flow of the timelike Galilean vector field K. Given a point $p \in M$, we can take an open neighborhood U_p in the leaf $\mathcal{F}_p$ of the foliation $\mathcal{F}$ induced by Ω passing through p and an open interval $I \subseteq \mathbb{R}$, $0 \in I$, such that the flow F_t is well defined and injective. Then, we may define the map

$$F : I \times U_p \longrightarrow F(I \times U_p) \subset M, \qquad (t, q) \longmapsto F_t(q),$$

which satisfies,

$$dF\mid_{(t,q)} (\partial_t, 0) = K(F_t(q)), \qquad dF\mid_{(t,q)} (0, v) = dF_t\mid_q (v),$$

for all $(t, q) \in I \times U_p$ and $v \in T_pU_p$. Identifying $\partial_t \equiv (\partial_t, 0)$, we obtain

$$F^*\Omega(\partial_t) = \Omega\big(dF(\partial_t)\big) = \Omega(K).$$

Moreover, taking into account the Leibnizian character of K,

$$F^*\Omega(0, v) = \Omega\big(dF\mid_{(s,q)} (0, v)\big) = \Omega\big(dF_s\mid_q (v)\big) = F_s^*\Omega\mid_q (v) = \Omega_q(v) = 0.$$

Therefore, $F^*\Omega = \Omega(K)\,dt$, and each level set $\{t = a\}$ corresponds with an open set of a leaf of $\mathcal{F}$. Furthermore, since K is Leibnizian, $K(\Omega(K)) = \Omega([K, K]) = 0$. Thus, there exists a function $\alpha \in C^\infty(U_p)$ such that $F^*\Omega = (\alpha \circ \pi_{U_p})dt$.

On the other hand, for any $v, w \in T_pU_p$, the second Leibnizian condition may be rewritten as

$$0 = L_K g\big(dF_s(v), dF_s(w)\big) =$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[g_{F_{s+\varepsilon}(q)}\big(dF_{s+\varepsilon}(v), dF_{s+\varepsilon}(w)\big) - g_{F_s(q)}\big(dF_s(v), dF_s(w)\big)\right] = \eta'(s),$$

where

$$\eta(s) := g\big(dF_s(v), dF_s(w)\big) = F^*g\mid_{(s,q)}\big((0,v),(0,w)\big).$$

Indeed, $L_{\partial_t}F^*g = 0$ and the induced metric on $I \times U_p$ is like (7), with $h = g_{|U_p}$.

Finally, denoting by $\nabla' = F^*\nabla$ the connection induced in $I \times U_p$ by F, we conclude that

$$\big(I \times U_p, (\alpha \circ \pi_{U_p})d\pi_I, \pi^*_{U_p} g_{|U_p}, \nabla'\big)$$

is a standard stationary Galilean spacetime and F is a Galilean diffeomorphism.

□

5 Standard decompositions of stationary Galilean spacetimes

We have previously seen in Proposition 12 that every stationary Galilean spacetime is locally a standard stationary Galilean spacetime. Here, our aim consists in looking for additional assumptions on the geometry of an stationary Galilean spacetime leading to a global splitting as a stardard stationary Galilean spacetime. A similar question has been yet discussed in relativistic settings (see for instance, [1] [2], [12] and [14]), i.e., under what conditions does a relativistic spacetime admit a global decomposition as a warped product space.

Theorem 13 *Let $\big(M, \Omega, g, \nabla\big)$ be a stationary Galilean spacetime with timelike Galilean vector field K. If the absolute clock satisfies $\Omega = \beta\, dT$ for some global functions $\beta, T \in C^\infty(M)$, $\beta > 0$ and the flow of K is well defined and onto in a domain $I \times \mathcal{F}_0$ for some interval $I \subseteq \mathbb{R}$ and some leaf of the foliation $\mathcal{F}_0$ induced by Ω, then the spacetime admits a global decomposition as a standard stationary Galilean spacetime.*

Proof. Note that the leaves of the foliation are the level sets of the global function T, i.e., $\mathcal{F}_t = \{q \in M : T(q) = t\}$. Taking the leaf $\mathcal{F}_0$ of the foliation induced by Ω and denoting by F_s the global flow of K, we can build the same map used in Proposition 12,

$$F : I \times \mathcal{F}_0 \longrightarrow M, \qquad (s,q) \longmapsto F_s(q),$$

which is onto by hypothesis. To prove the injectivity we only need to check that each integral curve of K intersects each leaf of $\mathcal{F}$ only once, which will be proved by contradiction.

Suppose that an integral curve of K, that we can denote by γ, intersects twice the same leaf $\mathcal{F}_b$, $b \in \mathbb{R}$. Hence, there are two values $s_1, s_2 \in I$, $s_1 < s_2$, such that $(T \circ \gamma)(s_1) = (T \circ \gamma)(s_2)$. Since the real function $T \circ \gamma : I \longrightarrow \mathbb{R}$ is smooth, Rolle's Theorem ensures the existence of $s^* \in (s_1, s_2)$ such that

$$\frac{d}{ds}(T \circ \gamma)(s^*) = 0, \qquad \Longleftrightarrow \qquad dT\big(K\big(\gamma(s^*)\big)\big) = 0,$$

which is in contradiction with the timelike character of K.

Using the bijectivity of F together with the local result shown in Proposition 12 we can conclude the proof.

□

Remark 14 Notice that when K is complete the hypothesis on the flow of K in Theorem 13 automatically holds.

Taking into account the previous Remark, we can assert

Corollary 15 *Let (M, Ω, g, ∇) be a stationary Galilean spacetime with timelike Galilean vector field K. If the absolute clock Ω is exact and K is complete, then M globally splits as a standard stationary Galilean spacetime.*

To end this section, we present a global splitting result when the spacetime is spatially compact, that is, when the leaves of the spacelike foliation are compact.

Theorem 16 *Let (M, Ω, g, ∇) be a stationary Galilean spacetime with $\Omega = \beta\, dT$ for some global functions $\beta, T \in C^\infty(M)$, $\beta > 0$. If the leaves of the foliation induced by Ω are compact, then M is a standard stationary Galilean spacetime.*

Proof. Let $F : \mathcal{D} \longrightarrow M$ be the maximal local flow of K and $p \in M$ a point located in a leaf $\mathcal{F}_p$ of the foliation induced by Ω. For each $q \in \mathcal{F}_p$, there exists a neighborhood $U_q \subset \mathcal{F}_p$ and an interval I_q such that F is defined in $I_q \times U_q$. Since $\mathcal{F}_p$ is compact, it can be written as $\mathcal{F}_p = \cup_{q \in A} U_q$, with A a finite set. Taking $I = \cap_{q \in A} I_q$, we conclude that the flow F is well defined in $I \times \mathcal{F}_p$.

Assume that $\bar{I} = (a, b)$ is the maximal interval where $F : \bar{I} \times \mathcal{F}_p \longrightarrow M$ is defined and let us see that $\bar{I}$ is also the maximal definition interval for each integral curve with initial value in $\mathcal{F}_p$. Suppose that there exists $p_0 \in \mathcal{F}$ such that $F(\cdot, p_0)$ is defined in $(a, b + \epsilon)$. Since all the leaves of the foliation are compact, we can take $\delta > 0$ such that $(-\delta, \delta) \times \mathcal{F}_{F_{(b,p_0)}} \subset \mathcal{D}$. Thus, it is possible to define the extension of the flow, which is a contradiction.

Finally, we will prove that $F(\bar{I} \times \mathcal{F}_p) = M$ reasoning by contradiction. Let us consider a point q in the complementary of $F(\bar{I} \times \mathcal{F}_p)$ and take the maximal interval J where $F : J \times \mathcal{F}_q \longrightarrow M$ is defined. The set $F(\bar{I} \times \mathcal{F}_p) \cap F(J \times \mathcal{F}_q)$ must be empty; if a point q_0 was in the intersection, then the integral curve of K passing through q_0 could be defined on a bigger interval than $\bar{I}$, contradicting the maximality of $\bar{I}$. Hence, taking into account that $F(\bar{I} \times \mathcal{F}_p)$ and its complementary are open in M and M is connected, we conclude that $F(\bar{I} \times \mathcal{F}_p) = M$. The same arguments of Proposition 12 serve to end the proof.

□

6 Completeness of free falling observers in stationary Galilean spacetimes

In this section we will analyze the geometrical conditions under which the inextensible geodesics in a stationary Galilean spacetime are complete. From a physical viewpoint, we are studying the hypothesis on the spacetime that ensure that free falling observers live forever.

First of all, given a timelike vector field $K \in \Gamma(TM)$, we may construct an auxiliary Riemannian metric on M, g_R, defined by

$$g_R(X,Y) = \Omega(X)\Omega(Y) + g(P^K(X), P^K(Y)), \qquad \forall X, Y \in \Gamma(TM), \tag{8}$$

where P^K is the projection on the spacelike leaves, $P^K(X) = X - \frac{\Omega(X)}{\Omega(K)}K$. From now on we will omit the brackets in the expression $P^K(X)$. Notice that with this metric the vector field $K/\Omega(K)$ is unitary and every spacelike vector is orthogonal to K. In addition, since the integral curves of $K/\Omega(K)$ are geodesics with the Levi-Civita connection of g_R, the completeness of g_R implies the completeness of $K/\Omega(K)$.

Theorem 17 *Let (M, Ω, g, ∇) be a stationary Galilean spacetime with symmetric connection and timelike Galilean vector field K. If the auxiliary metric g_R is complete and the gravitational field $\mathcal{G}$ associated to $K/\Omega(K)$ is bounded on any spacelike leaf $\mathcal{F}$ of* $\mathrm{An}(\Omega)$*, i.e.,*

$$\sup_{\mathcal{F}} (g(\mathcal{G}, \mathcal{G})) \leq L^2 \qquad L > 0, \tag{9}$$

then each inextensible geodesic is complete.

Proof. Since ∇ is symmetric, $d\Omega = 0$. Moreover, using the Leibnizian character of K we may suppose without loss of generality that $\Omega(K) = 1$, i.e., K is a field of observers. Now, let $\gamma : [0,1) \longrightarrow M$ be a geodesic in M. From [19, Lemma 5.8] (see also [5, Sec. 2]), it is enough to prove that the g_R-length of γ is bounded.

First, from the Galilean character of M, we know that

$$0 = \Omega\Big(\frac{D\gamma'}{dt}\Big) = \gamma'(\Omega(\gamma')),$$

where $\frac{D}{dt}$ is the covariant derivative associated to ∇. Thus, $\Omega(\gamma')$ is constant along γ and can be denoted by $\lambda = \Omega(\gamma')$. Taking this into account and using (8) we get

$$\|\gamma'\|_R^2 = g_R(\gamma', \gamma') = \lambda^2 + g(P^K\gamma', P^K\gamma'). \tag{10}$$

Moreover, from the Galilean character of ∇,

$$\frac{d}{dt}g(P^K\gamma', P^K\gamma') = 2\, g\Big(\frac{D}{dt}P^K\gamma', P^K\gamma'\Big) = -2\lambda\, g\Big(\frac{DK}{dt}, P^K\gamma'\Big). \tag{11}$$

Let us consider $Y \in \Gamma(TM)$ such that $Y \circ \gamma = \gamma'$. Hence, for each $V \in \Gamma(\mathrm{An}(\Omega))$, we compute making use of equation (1),

$$2\,g\Big(\frac{DK}{dt},V\Big) = 2\,g(\nabla_Y K, V) = K\big(g(P^K Y, V)\big) + 2\lambda\, g(\mathcal{G}, V) +$$
$$+2\omega(K)(P^K Y, V) - \big(g([K, P^K Y], V) + g([K,V], P^K Y)\big) =$$
$$= 2\lambda\, g(\mathcal{G}, V) + 2\,\omega(K)(P^K Y, V),$$

where we have used the Leibnizian character of K in the last equality. Consequently, inserting $V = P^K\gamma'$ in the latter formula, equation (11) yields

$$\frac{d}{dt} g(P^K\gamma', P^K\gamma') = -2\,\lambda\, g\Big(\lambda\, g(\mathcal{G}, P^K\gamma') + \omega(K)(P^K\gamma', P^K\gamma')\Big) = -2\,\lambda^2 g(\mathcal{G}, P^K\gamma').$$

Furthermore, since $K(g(\mathcal{G},\mathcal{G})) = 2g([K,\mathcal{G}],\mathcal{G}) = 0$, the function $g(\mathcal{G},\mathcal{G})$ is constant along the integral curves of K. In fact, any two leaves $\mathcal{F}_1, \mathcal{F}_2$ of Ω can be connected by means of a curve γ such that $\Omega(\gamma') \geq 0$, so the d-flow of $K/\Omega(K)$ will yield an isometry between $\mathcal{F}_1$ and $\mathcal{F}_2$, being $d = \int \Omega\big(\gamma'(s)\big) ds$. Hence, if the bound (9) holds for one leaf then it will hold for all the leaves.

Therefore, using (9) we deduce

$$g(P^K\gamma', P^K\gamma') = -2\,\lambda^2 \int_0^t g(\mathcal{G}, P^K\gamma')\, ds \leq 2\,\lambda^2 L \int_0^t \big(g(P^K\gamma', P^K\gamma')\big)^{1/2}\, ds, \tag{12}$$

where we have used the Cauchy-Schwarz inequality. Replacing this in equation (10) we obtain

$$\|\gamma'\|_R^2 \leq \lambda^2 + 2\,\lambda^2 L \int_0^t \|\gamma'\|_R\, ds,$$

or equivalently,

$$\frac{\|\gamma'\|_R}{\sqrt{1 + 2L\int_0^t \|\gamma'\|_R\, ds}} \leq \lambda.$$

Integrating this expression in $[0,t]$, $0 < t < 1$, we obtain

$$\left(1 + 2L \int_0^t \|\gamma'\|_R\, ds\right)^{1/2} \leq \lambda\, L\, t,$$

and we conclude that $\int_0^t \|\gamma'\|_R\, ds \leq B$, for some positive constant B.

□

As a consequence of Theorem 17 we obtain for the compact case the following result.

Corollary 18 *Compact stationary Galilean spacetimes with symmetric connection are geodesically complete.*

Furthermore, for standard stationary Galilean spacetimes we have

Corollary 19 *Let $(M = \mathbb{R} \times S, \Omega = \alpha dt, g = \pi_S^* h, \nabla)$ be a standard stationary Galilean spacetime with α constant and denote by $\mathcal{G} := \nabla_{\partial_t}\partial_t$. If (S,h) is complete and the function $h(d\pi_S(\mathcal{G}), d\pi_S(\mathcal{G}))$ is bounded on S, then M is geodesically complete.*

Proof. From [19, Prop. 7.40], the auxiliary metric g_R is complete. In addition, Proposition 9 ensures that the connection is symmetric and Theorem 17 applies.

□

Example 20 For instance, if we denote by $(\mathbb{S}^3, g_{\mathbb{S}^3})$ the unit round 3-sphere with its usual Riemannian metric, Corollary 19 ensures the geodesic completeness of the standard stationary spacetime $(\mathbb{R} \times \mathbb{S}^3, \Omega = dt, g = \pi^*_{\mathbb{S}^3} g_{\mathbb{S}^3}, \nabla)$, where ∇ is necessarily symmetric by Proposition 9.

Moreover, combining Corollary 15 and Theorem 17, we can enunciate

Corollary 21 *Let (M, Ω, g, ∇) be a simply connected stationary Galilean spacetime with symmetric connection and complete Galilean vector field K. Let $\mathcal{F}_0$ be a leaf of the foliation induced by Ω. If the metric $g_{|\mathcal{F}_0}$ is complete and $g_{|\mathcal{F}_0}(\mathcal{G}, \mathcal{G})$ is bounded, then the spacetime M is geodesically complete.*

Finally, from Theorem 16 and Corollary 19 we obtain

Corollary 22 *Let $\big(M, \Omega, g, \nabla\big)$ be a stationary Galilean spacetime with $\Omega = \beta\, dT$ for some global functions $\beta, T \in C^\infty(M)$, $\beta > 0$ and symmetric connection. If the leaves of the foliation induced by Ω are compact, then the spacetime M is geodesically complete.*

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