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SOME NEW RESULTS ON THE k-TUPLE DOMINATION NUMBER OF GRAPHS

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Abstract. Let $k \geq 1$ be an integer and G be a graph of minimum degree $\delta(G) \geq k-1$. A set $D \subseteq V(G)$ is said to be a k-tuple dominating set of G if $|N[v] \cap D| \geq k$ for every vertex $v \in V(G)$, where $N[v]$ represents the closed neighbourhood of vertex v. The minimum cardinality among all k-tuple dominating sets is the k-tuple domination number of G. In this paper, we continue with the study of this classical domination parameter in graphs. In particular, we provide some relationships that exist between the k-tuple domination number and other classical parameters, like the multiple domination parameters, the independence number, the diameter, the order and the maximum degree. Also, we show some classes of graphs for which these relationships are achieved.

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1. Introduction

Throughout this article we consider G as a simple graph with vertex set $V(G)$. Given a vertex v of G, $N(v)$ and $N[v]$ represent the *open neighbourhood* and the *closed neighbourhood* of v, respectively. Given a vertex $v \in V(G)$ and a set $D \subseteq V(G)$, we denote by $\deg_D(v) = |N(v) \cap D|$ the number of vertices in D adjacent to v and let $\deg_D[v] = |N[v] \cap D|$. The values $\delta(G) = \min\{\deg_{V(G)}(x) : x \in V(G)\}$ and $\Delta(G) = \max\{\deg_{V(G)}(x) : x \in V(G)\}$ denote the *minimum* and *maximum degrees* of G, respectively. A graph is *claw-free* if and only if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph. Other definitions not given here can be found in the book [1].

In 1985, Fink and Jacobson [2, 3] introduced the k-domination in graphs as an extension of the concept of domination in graphs. A set $D \subseteq V(G)$ is said to be a k-*dominating set* of G if $\deg_D(v) \geq k$ for every $v \in V(G) \setminus D$. Notice that the 1-dominating set of G is the same as the classical dominating set of G. The k-*domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality among all k-dominating sets of G. We define a $\gamma_k(G)$-set as a k-dominating set of G with cardinality $\gamma_k(G)$. The same agreement will be assumed for optimal parameters associated to other characteristic sets defined in the article.

More than 10 years later, and in two different papers (published in 1996 and 2000, respectively), Harary and Haynes [4, 5] introduced the concept of k-tuple domination in graphs. Given a graph G and a positive integer $k \leq \delta(G)+1$, a set $D \subseteq V(G)$ is said to be a k-*tuple dominating set* of G if $\deg_D[v] \geq k$ for every $v \in V(G)$.

Keywords. k-domination, k-tuple domination, double domination.

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Observe that the 1-tuple dominating set of G is the same as the dominating set of G. The *k-tuple domination number* of G, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality among all k-tuple dominating sets of G. For a comprehensive survey on k-domination and k-tuple domination in graphs, we suggest the chapter [6] due to Hansberg and Volkmann. In addition, some recent results on these parameters can be found in [7–11].

In [12], Liao and Chang proved that the problem of deciding if a given graph G has a k-tuple dominating set of cardinality $\gamma_{\times k}(G)$ is NP-hard, even for bipartite and split graphs. This suggests finding the k-tuple domination number for special classes of graphs, obtaining tight bounds, as well as providing relationships between this parameter and other domination invariants in graphs. In this paper, we continue with the study of this classical domination parameter in graphs. In particular, we center our attention on these last two goals.

2. Relationships with other domination parameters

Relationships between different parameters corresponding to multiple domination have attracted the attention of several researches in the last few decades, and a high number of significant contributions are nowadays well known.

Recently, Hansberg and Volkmann [6] put into context all relevant relationships concerning k-tuple domination (with emphasis in the case $k = 2$) that have been found up to 2020. Subsequently, Cabrera-Martínez [8,13] obtained new results in this direction. In particular, the following theorem solved an open problem posed in [6].

Theorem 2.1. [8] *Let $k \geq 2$ be an integer. For any graph G with $\delta(G) \geq k-1$,*

$$\gamma_{\times k}(G) \leq k\gamma_k(G) - (k-1)^2.$$

The next result provides a new upper bound for the k-tuple domination number of a graph G in terms of the k-domination number and the k'-tuple domination number ($k' \in \{1, \ldots, k-1\}$).

Theorem 2.2. *Let k', k be two integers such that $k > k' \geq 1$. For any graph G with $\delta(G) \geq k-1$,*

$$\gamma_{\times k}(G) \leq \gamma_{\times k'}(G) + (k-k')\gamma_k(G).$$

Proof. Let S be a $\gamma_{\times k'}(G)$-set, D a $\gamma_k(G)$-set and $D_0 = \{v \in D : \deg_{D\cup S}(v) < k-1\}$. As S is a k'-tuple dominating set of G, we deduce that

$$\sum_{v\in D_0\setminus S} \deg_{D\cup S}(v) \geq k'|D_0 \setminus S| \quad \text{and} \quad \sum_{v\in D_0\cap S} \deg_{D\cup S}(v) \geq (k'-1)|D_0 \cap S|,$$

which implies that

$$\sum_{v\in D_0} \deg_{D\cup S}(v) \geq (k'-1)|D_0| + |D_0 \setminus S|. \tag{2.1}$$

Moreover, as $k > k'$, we deduce the following inequality.

$$(k-1)|D \setminus D_0| \geq (k'-1)|D \setminus D_0| + |D \setminus D_0| \geq (k'-1)|D \setminus D_0| + |D \setminus (D_0 \cup S)|. \tag{2.2}$$

From inequalities (2.1) and (2.2) we deduce the following inequality.

$$\sum_{v\in D_0} \deg_{D\cup S}(v) + (k-1)|D \setminus D_0| \geq (k'-1)|D| + |D \setminus S|. \tag{2.3}$$

Now, we define $W' \subseteq V(G)$ as a set of minimum cardinality among all supersets W of $D \cup S$ such that $\deg_W(v) \geq k-1$ for every $v \in D$. Since $\deg_{D\cup S}(x) \geq k-1$ for every $x \in D \setminus D_0$, the condition on W is

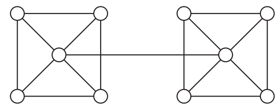

FIGURE 1. A graph G with $\gamma_{\times 2}(G) = \gamma_2(G) + \gamma(G)$.

equivalent to that every vertex $v \in D_0$ has at least $k - 1 - \deg_{D\cup S}(v)$ neighbours in $W \setminus (D \cup S)$. Hence, by the minimality of W' and inequality (2.3), we deduce that

$$\begin{aligned}
|W' \setminus (D \cup S)| &\leq |D_0|(k-1) - \sum_{v \in D_0} \deg_{D\cup S}(v) \\
&= |D|(k-1) - \left(\sum_{v \in D_0} \deg_{D\cup S}(v) + |D \setminus D_0|(k-1)\right) \\
&\leq |D|(k-1) - ((k'-1)|D| + |D \setminus S|) \\
&= (k-k')|D| - |D \setminus S|.
\end{aligned}$$

Moreover, it is easy to check that W' is a k-tuple dominating set of G because each vertex in $V(G) \setminus D$ is dominated k times by vertices of $D \subseteq W'$ (recall that D is a k-dominating set of G) and the construction of W' ensures that each vertex in D is dominated k times by vertices of W'. Therefore,

$$\begin{aligned}
\gamma_{\times k}(G) &\leq |W'| \\
&= |S| + |D \setminus S| + |W' \setminus (D \cup S)| \\
&\leq |S| + |D \setminus S| + ((k-k')|D| - |D \setminus S|) \\
&= |S| + (k-k')|D| \\
&= \gamma_{\times k'}(G) + (k-k')\gamma_k(G),
\end{aligned}$$

which completes the proof. □

The bound given in Theorem 2.2 is tight for certain values of k. For instance, when $k = k' + 1 = 2$, then the bound is achieved for the graph G given in Figure 1 since $\gamma_{\times 2}(G) = 6$, $\gamma_2(G) = 4$ and $\gamma(G) = 2$. Moreover, if $k = k' + 1$ for any integer $k \geq 2$, then the bound is achieved for the join graph $G_{k,r}$ defined after Corollary 2.4.

The following result, which is a direct consequence of Theorem 2.2 (considering the particular case where $k' = 1$), improves the upper bound given in Theorem 2.1 whenever $\gamma_k(G) > \gamma(G) + (k-1)^2$.

Corollary 2.3. *Let $k \geq 2$ be an integer. For any graph G with $\delta(G) \geq k - 1$,*

$$\gamma_{\times k}(G) \leq (k-1)\gamma_k(G) + \gamma(G).$$

The corollary above for the case $k = 2$ was given in [14], and the authors showed that the bound is achieved for the graph given in Figure 1.

We continue with another consequence derived from Theorem 2.2. For this purpose, we now consider the particular case where $k' = k - 1$.

Corollary 2.4. *Let $k \geq 2$ be an integer. For any graph G with $\delta(G) \geq k - 1$,*

$$\gamma_{\times k}(G) \leq \gamma_{\times (k-1)}(G) + \gamma_k(G).$$

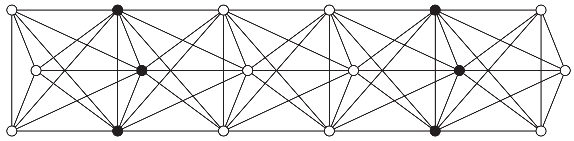

FIGURE 2. The set of black vertices forms a $\gamma_{\times 3}(P_6 \circ K_3)$-set.

For any integers $k, r \in \mathbb{Z}$ with $r \geq k \geq 2$, let $G_{k,r}$ be the join graph obtained from the complete graph K_{k-1} and the trivial graph N_r, *i.e.*, $G_{k,r} = K_{k-1} \vee N_r$. Observe that $\gamma_{\times k}(G_{k,r}) = |V(G_{k,r})| = k-1+r$, $\gamma_{\times(k-1)}(G_{k,r}) = k-1$ and $\gamma_k(G_{k,r}) = r$. Therefore, for these graphs the bound given in Corollary 2.4 is tight.

In 2001, Favaron *et al.* [15] showed that $\gamma_{\times k}(G) \geq \gamma_{k'}(G) + k - k'$ for any graph G with $\delta(G) \geq k > k' \geq 1$. The bound given in the following result improves the previous one whenever $\mathrm{diam}(G) \geq 5$ ($\mathrm{diam}(G)$ represents the *diameter* of a connected graph G).

Theorem 2.5. *Let k', k be two integers such that $k > k' \geq 1$. For any connected graph G with $\delta(G) \geq k$,*

$$\gamma_{\times k}(G) \geq \gamma_{k'}(G) + (k-k')\left\lceil \frac{diam(G)+1}{5} \right\rceil.$$

Proof. Let D be a $\gamma_{\times k}(G)$-set. Let $P = v_0 v_1 \cdots v_r$ be a diametrical path of G (in this case, $r = \mathrm{diam}(G)$) and $X = \{v_0, v_5, \ldots, v_{5\lfloor r/5 \rfloor}\}$. Now, let D' be the subset of $N[X] \cap D$ such that $|D'| = (k-k')|X|$ and $\deg_{D'}[x] = k-k'$ for every $x \in X$. By the definitions of D and X, it follows that D' is well defined. Moreover, and by the definition of X, if $x, y \in X$ (with $x \neq y$), then $N[u_x] \cap N[u_y] = \emptyset$ for any $u_x \in N[x]$ and any $u_y \in N[y]$. We claim that $W = D \setminus D'$ is a k'-dominating set of G. Let $v \in V(G) \setminus W$. Since $D' \subseteq N[X]$ and $\deg_{D'}[x] = k - k'$ for every $x \in X$, it follows that $\deg_{D'}[v] \leq k - k'$. Now, we analyse the next two cases.

Case 1: $v \in V(G) \setminus D$. In this case, we have that $\deg_D(v) \geq k$ and $\deg_{D'}(v) \leq k - k'$. This implies that $\deg_W(v) \geq \deg_D(v) - \deg_{D'}(v) \geq k - (k-k') = k'$, as required.

Case 2: $v \in D'$. In this case, we have that $\deg_D(v) \geq k-1$ and $\deg_{D'}(v) \leq k - k' - 1$. Hence, $\deg_W(v) \geq \deg_D(v) - \deg_{D'}(v) \geq (k-1) - (k-k'-1) = k'$, as required.

From the both cases above, we deduce that W is a k'-dominating set of G, which implies that

$$\gamma_{k'}(G) \leq |W| = |D| - |D'| = |D| - (k-k')|X| = \gamma_{\times k}(G) - (k-k')\left\lceil (\mathrm{diam}(G)+1)/5 \right\rceil.$$

Therefore, the proof is complete. □

The *lexicographic product* of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$ whose vertex set is $V(G_1 \circ G_2) = V(G_1) \times V(G_2)$ and $(u,v)(x,y) \in E(G_1 \circ G_2)$ if and only if $ux \in E(G_1)$ or $u = x$ and $vy \in E(G_2)$. For instance, in Figure 2 we have the graph $P_6 \circ K_3$.

The lower bound given in Theorem 2.5 is tight due to $\gamma_{\times k}(P_6 \circ K_k) = 2k$, $\gamma_{k'}(P_6 \circ K_k) = 2k'$ and $\mathrm{diam}(P_6 \circ K_k) = 5$ for any integers k, k' with $k > k' \geq 1$.

3. Bounds in terms of order, independence number and maximum degree

We begin this section by restating the following well-known trivial upper bounds.

Theorem 3.1. [15] *For any graph G with $\delta(G) \geq k-1$,*

$$\gamma_{\times k}(G) \leq n(G) - \delta(G) + k - 1 \leq n(G).$$

Klasing and Laforest [16] obtained the lower bound $\gamma_{\times k}(G) \geq k\alpha(G)/2$ ($\alpha(G)$ represents the independence number of G), assuming that G is a claw-free graph with $\delta(G) \geq k-1$. The following result relates the k-tuple domination number with the independence number, but in the opposite sense to that shown in the previous bound.

Proposition 3.2. *Let $k \geq 2$ be an integer. For any claw-free graph G of order $n(G)$ and $\delta(G) \geq k+1$,*

$$\gamma_{\times k}(G) \leq n(G) - \alpha(G).$$

Proof. Let I be an $\alpha(G)$-set. We proceed to prove that $W = V(G) \setminus I$ is a k-tuple dominating set of G. It is clear that $\deg_W(x) \geq \delta(G) > k$ for every vertex $x \in I$. Let $v \in W$. If $\deg_W(v) \leq k-2$, then $\deg_I(v) \geq 3$ as $\delta(G) \geq k+1$, which contradicts the fact that G is claw-free. Hence, $\deg_W(v) \geq k-1$, as required. Therefore, W is a k-tuple dominating set of G, and so, $\gamma_{\times k}(G) \leq |W| = n(G) - \alpha(G)$. □

The bound above is tight for certain values of k. For instance, when $k=2$, the bound is achieved for the join graph $G = K_1 \vee C_4$ since $\gamma_{\times 2}(G) = 3$, $\alpha(G) = 2$ and $n(G) = 5$.

The next result provides a new upper bound on $\gamma_{\times k}(G)$ in terms of the order and the maximum degree of a graph G with minimum degree $\delta(G) \geq k$.

Theorem 3.3. *Let $k \geq 2$ be an integer. For any graph G with $\delta(G) \geq k$,*

$$\gamma_{\times k}(G) \leq \frac{k\Delta(G)}{k\Delta(G)+1}\, n(G).$$

Proof. Let D be a $\gamma_{\times k}(G)$-set. We now consider the next subsets of vertices.

$$D^* = \{x \in D : N(x) \subseteq D\} \qquad \text{and} \qquad D^= = \{x \in D : \deg_D(x) = k-1\}.$$

Notice that $D^* \cap D^= = \emptyset$ and $N(x) \cap D^= \neq \emptyset$ for any vertex $x \in D^*$. This implies that $|D^*| \leq (k-1)|D^=| \leq (k-1)|D \setminus D^*|$, and as a consequence, $|D^*| \leq \frac{k-1}{k}|D|$.

Moreover, we notice that $V(G) \setminus D$ is a dominating set of $G - D^*$. This implies that

$$\frac{n(G) - |D^*|}{\Delta(G)+1} \leq \frac{n(G) - |D^*|}{\Delta(G - D^*)+1} \leq |V(G) \setminus D| = n(G) - |D|,$$

and as a consequence, $|D^*| \geq |D|(\Delta(G)+1) - n(G)\Delta(G)$. Combining the two previous bounds, we obtain that

$$|D|(\Delta(G)+1) - n(G)\Delta(G) \leq \frac{k-1}{k}|D|,$$

and as a consequence, $|D| \leq \frac{k\Delta(G)}{k\Delta(G)+1}\, n(G)$, which completes the proof. □

In [5], Harary and Haynes showed that $\gamma_{\times k}(G) \geq \frac{kn(G)}{\Delta(G)+1}$ for any graph G with $\delta(G) \geq k-1$. The following result provides a partial refinement of the bound above, which is evidenced only for the graphs G satisfying $(\Delta(G)+1)|V_\Delta(G)| < kn(G)$, where $V_\Delta(G) = \{v \in V(G) : \deg_{V(G)}(v) = \Delta(G)\}$.

Theorem 3.4. *Let $k \geq 2$ be an integer. Let G be a graph and let $V_\Delta(G) = \{v \in V(G) : \deg_{V(G)}(v) = \Delta(G)\}$. If $\delta(G) \geq k-1$, then*

$$\gamma_{\times k}(G) \geq \frac{kn(G) - |V_\Delta(G)|}{\Delta(G)}.$$

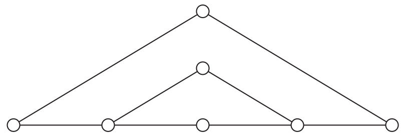

FIGURE 3. A graph G with $\gamma_{\times 2}(G) = 4$.

Proof. Given two sets $D_1, D_2 \subseteq V(G)$, let $E(D_1, D_2) = \{uv \in E(G) : u \in D_1 \text{ and } v \in D_2\}$. Let D be a $\gamma_{\times k}(G)$-set. Hence,

$$\begin{aligned} k(n(G) - |D|) &\le |E(V(G) \setminus D, D)| \\ &\le \sum_{v \in D} (\deg_{V(G)}(v) - \deg_D(v)) \\ &\le |D \cap V_\Delta(G)|(\Delta(G) - k + 1) + |D \setminus V_\Delta(G)|(\Delta(G) - k), \end{aligned}$$

which implies that $|D| \ge \frac{kn(G) - |D \cap V_\Delta(G)|}{\Delta(G)} \ge \frac{kn(G) - |V_\Delta(G)|}{\Delta(G)}$. Therefore, the proof is complete. □

The bound above is tight for certain values of k. For instance, when $k = 2$ then the bound is achieved for the graph G given in Figure 3. For this graph, we have that $\gamma_{\times 2}(G) = 4$, $|V(G)| = 7$, $\Delta(G) = 3$ and $|V_\Delta(G)| = 2$, which implies that

$$\gamma_{\times 2}(G) = 4 = \frac{2|V(G)| - |V_\Delta(G)|}{\Delta(G)} > \frac{2|V(G)|}{\Delta(G) + 1}.$$

Moreover, the bound is also achieved for any join graph $G = K_k \vee H$, obtained from the complete graph K_k and any graph H of order $|V(H)| = k \ge 2$ and $\Delta(H) < k - 1$. For this join graph G, we have that $\gamma_{\times k}(G) = |V_\Delta(G)| = k$, $|V(G)| = 2k$ and $\Delta(G) = 2k - 1$.

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