Complete spacelike hypersurfaces on symmetric spacetimes

Alma L. Albujer

Departamento de Matemáticas, Edificio Albert Einstein
Universidad de Córdoba, Campus de Rabanales,
14071 Córdoba, Spain

E-mail: aalbujer@uco.es

Jónatan Herrera

Departamento de Matemáticas, Edificio Albert Einstein
Universidad de Córdoba, Campus de Rabanales,
14071 Córdoba, Spain

E-mail: jherrera@uco.es

Rafael M. Rubio

Departamento de Matemáticas, Edificio Albert Einstein
Universidad de Córdoba, Campus de Rabanales,
14071 Córdoba, Spain

E-mail: ma1rurur@uco.es

Dezember 2019

Abstract. A Lorentz manifold (M, g) is said to be a conformally stationary spacetime if it is endowed with a globally defined conformal timelike vector field K, whereas it is a pp-wave when there is a globally defined parallel lightlike vector field K on M. The study of rigidity and non-existence results for spacelike hypersurfaces with constant mean curvature in these spaces has been considered in the last years by several authors. In this note we unify some known techniques and results related to both problems by considering spacelike hypersurfaces in a spacetime (M, g) endowed with a globally defined conformal *causal* vector field. This wide family of Lorentz manifolds not only includes previous cases, but it also includes new families of spacetimes.

Keywords: conformally stationary spacetime, pp-wave, conformal vector field, constant mean curvature, parabolicity.

Mathematical Subjet Classification: Primary: 53C42; Secondary: 53C50, 83C20.

Submitted to: *Class. Quantum Grav.*

1. Introduction

The concept of symmetry is essential in Physics. In General Relativity a symmetry in a spacetime, or an infinitesimal symmetry, is given by the existence of a global Killing vector field, or more generally a conformal vector field. In fact, as an important problem in General Relativity is the search of exact solutions of Einstein's equations, the assumption of the existence of an (a priori) infinitesimal symmetry on the spacetime constitutes an effective method in the search for such solutions, see for instance [8, 9]. Observe that different causal characters can be assumed for the infinitesimal symmetry of the spacetime. In fact, the timelike choice leads to the class of conformally stationary spacetimes (see [1]). Otherwise, when considering a parallel lightlike vector field we get the family of pp-wave spacetimes, (see for instance [15, Section 24.5]). However, the same causal character for the infinitesimal symmetry is not always assumed, as happens with spacetimes like the Schwarzschild, the Reissner-Nordströms, or even the Kerr spacetime (in this last one the Killing vector field becomes even spacelike in some regions, see [10, Chapter 5] for details).

Another possibility to look for exact solutions to the Einstein's equations is to generate new solutions from one already known, which can be made via a conformal transformation of the spacetime. This technique was first considered by Brinkmann in [4], so we will refer to it as the Brinkmann program (see Remark 3.1). In particular, it seems natural to study spacetimes whose metric is conformal to the metric of a pp-wave.

Furthermore, spacelike hypersurfaces of constant mean curvature and in particular maximal hypersurfaces, that is spacelike hypersurfaces whose mean curvature is identically zero, are an important subject of study in General Relativity, since they play a relevant role for the initial value problem as initial hypersurfaces where the constraint equations can be split into a linear system and a nonlinear elliptic equation (see [7] and references therein). A summary of other reasons justifying the study of CMC spacelike hypersurfaces can be found in [11].

Alías, Romero and Sánchez considered in [1] compact, constant mean curvature, spacelike hypersurfaces immersed in a conformally stationary spacetime, and obtained several uniqueness and non-existence results making use several integral formulae. More recently Camargo, Caminha, de Lima and Velásquez extended in [5] the study to the case of complete, not necessarily compact, hypersurfaces in conformally stationary spacetimes, and in particular in general Robertson-Walker spacetimes. Some years ago, Pelegrín, Romero and Rubio have considered in [13] compact spacelike hypersurfaces immersed in a pp-wave.

In this short note, we extend the above results to the case of complete parabolic spacelike hypersurfaces of constant mean curvature immersed in a wide class of spacetimes, those admitting a conformal causal vector field. Moreover, we also obtain results for the non-parabolic case under some mild regularity hypotheses. Finally, we analyse the restricted case of *closed* conformal vector fields, where our results can be sharpened even further.

2. Setting up

Along this note, let (M, g) be an $(n+1)$-dimensional $(n \geq 2)$ spacetime admitting a globally defined causal conformal field $K \in \mathfrak{X}(M)$, where as usual by causal we mean that K do not vanish anywhere and $g(K, K) \leq 0$. Let us recall that K is conformal if and only if $\mathcal{L}_K g = 2\varrho g$ for a certain function $\varrho \in \mathcal{C}^\infty(M)$, where $\mathcal{L}$ denotes the Lie derivative. As it has been remarked before, in the case where K is everywhere timelike (M, g) is known in the literature as a *conformally stationary spacetime*, and in the case where K is a parallel lightlike vector field (M, g) is known, following [15, Section 24.5], as a *pp-wave spacetime*

Let $x : S^m \to M^{n+1}$, $m < n+1$, be an isometrically immersed spacelike submanifold in M. As usual we will identify any point $p \in S$ with its image $x(p) \in M$, any vector $v \in T_pS$ with $dx_p(v) \in T_{x(p)}M$ and we will denote also by g the metric induced on S from (M, g).

Let us denote by $\mathfrak{X}_M(S)$ the set of vector fields on M along S. Observe that any vector field $V \in \mathfrak{X}_M(S)$ admits a unique decomposition

$$V = V^\top + V^\perp,$$

such that $V^\top \in \mathfrak{X}(S)$ and $V^\perp \in \mathfrak{X}^\perp(S)$, i.e. for any $p \in S$ it holds $V_p \in T_pS$ and $V_p^\perp \in T_p^\perp S$. Besides that, given any $V \in \mathfrak{X}_M(S)$ we can define the operator

$$\mathrm{div}_S(V) := \sum_i g\left(\overline{\nabla}_{E_i} V, E_i\right), \tag{1}$$

where $\overline{\nabla}$ stands for the Levi-Civita connection on M and $\{E_i\}_{i=1}^m$ is a local orthonormal frame of S, i.e., each E_i is a local section of the vector bundle TS. It is worth pointing out that the operator div_S is well-defined since it is independent of the chosen orthonormal frame. Let us also recall that for any $X \in \mathfrak{X}(S)$ and $\xi \in \mathfrak{X}^\perp(S)$ the classical Weingarten formulae for spacelike submanifolds asserts that

$$\overline{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi, \tag{2}$$

where $\nabla_X^\perp \xi = (\overline{\nabla}_X \xi)^\perp$ and $A_\xi : TS \to TS$ denotes the shape operator of S with respect to ξ. Thus, considering $\{N_j\}_{j=1}^{n+1-m} \subset T^\perp S$ a local orthonormal frame of local sections of the normal vector bundle of S in M, from (1) and (2) we easily obtain

$$\mathrm{div}_S(V^\perp) = \sum_{i,j} \epsilon_j\, g\left(V^\perp, N_j\right)\, g\left(\overline{\nabla}_{E_i} N_j, E_i\right) = -g\left(V, m\vec{H}\right),$$

where $\epsilon_j = g(N_j, N_j)$ and $\vec{H}$ is the mean curvature vector field of x. Hence,

$$\mathrm{div}\left(V^\top\right) = \mathrm{div}_S(V) + g\left(V, m\vec{H}\right),$$

div being the divergence operator on S.

In particular, if the submanifold S is closed (i.e. compact without boundary), making use of the Gauss theorem we obtain the following integral formula,

$$\int_S \left\{ \mathrm{div}_S(X) + g\left(X, m\vec{H}\right) \right\} dV_g = 0 \,. \tag{3}$$

where dV_g is the Riemannian volume element of (S, g).

From now on, we will restrict ourselves to the case where S is in fact an (immersed) spacelike hypersurface in M. Under this assumption, there exists a unique timelike normal vector field N globally defined on S with the same time-orientation as K. We will refer to N as the future-pointing Gauss map of S. As it is well-known, the mean curvature function of S with respect to N is given by $H = -\frac{1}{n}\mathrm{tr}(A)$, where $A = A_N$. This choice on the sign of H guarantees that $\vec{H}$ and N have the same time-orientation wherever $H(p) > 0$. Then, taking $X = K$ and recalling that K is conformal, the integral formula (3) reads

$$\int_S \{\varrho + Hg(K, N)\} \, dV_g = 0 \,. \tag{4}$$

We can now state a first non-existence result, which is a direct generalization of [1, Proposition 2.1],

Proposition 2.1. *Let (M, g) be a spacetime endowed with a globally defined conformal causal field K with non-positive (non-negative, respectively) conformal factor ϱ, then there do not exist any closed spacelike hypersurface S in M with $H > 0$ ($H < 0$, respectively).*

Proof. The proof follows directly from (4) by observing that $g(K, N) < 0$ on S. □

As an immediate consequence, we get the following result,

Corollary 2.2. *Let (M, g) be a spacetime endowed with a globally defined Killing causal vector field, then the only closed spacelike hypersurfaces S in M such that its mean curvature function does not change sign are maximal. In particular, there are no closed spacelike hypersurfaces with signed mean curvature in a stationary spacetime or in a pp-wave but the maximal ones.*

From now on, our aim is to generalize our results to a wider family of not necessarily compact spacelike hypersurfaces. In order to do it, let us now focus on the distinguished function $\Phi \in \mathcal{C}^\infty(S)$ given by

$$\Phi(p) = -g(K(p), N(p)).$$

Because of our choice of N it holds $\Phi \geq 0$ on S. Let us observe that in the case where K is timelike Φ measures the hyperbolic cosinus of the hyperbolic angle determined by N and K. Furthermore, we can obtain the following expression for its Laplacian, $\Delta\Phi$,

Lemma 2.3. *Let (M, g) be a spacetime endowed with a globally defined conformal causal field K, S a spacelike hypersurface in M and N its future Gauss map. Then, the function $\Phi = -g(K, N)$ satisfies*

$$\Delta\Phi = -\Phi\left(\operatorname{tr}(A^2) + \overline{\operatorname{Ric}}(N, N)\right) + ng(N, \overline{\nabla}\varrho) - ng(K^\top, \nabla H) - n\varrho H, \tag{5}$$

where $\overline{\operatorname{Ric}}$ and $\overline{\nabla}$ stand for the Ricci operator and the gradient in M, respectively.

The proof of Lemma 2.3 is a tedious but straightforward computation which follows from the Codazzi equation and the fact of K being conformal. We will omit the proof since it is analogous to the one in the case of (M, g) being a conformally stationary spacetime (see for instance [1] and [2, Proposition 3.1]) or in the case where it is a pp-wave (see [13, Section 3]). In order to adapt the proof to our more general situation, we only have to realize that it is enough by asking K to be causal. In fact, it does not necessarily keep the same causal character along S.

3. On the geometry of complete spacelike hypersurfaces

Let us show in this section several results for not necessarily closed spacelike hypersurfaces in (M, g), obtained mainly as an application of (5). On the one hand, we will assume that the spacetime obeys the timelike convergence condition (TCC). Let us recall that a spacetime satisfies TCC when the Ricci tensor acting on timelike vector fields is always non-negative, i.e. $\overline{\operatorname{Ric}}(Z, Z) \geq 0$ for any timelike field $Z \in \mathfrak{X}(M)$. From a physics point of view TCC is a relevant energy condition in General Relativity, which is traduced in the fact that, on average, gravity attracts. On the other hand, we will also ask the hypersurface to be parabolic. A complete Riemannian manifold is said to be parabolic if any subharmonic function bounded from above is necessarily constant. Let us observe that compact manifolds are a particular case of parabolic ones.

Theorem 3.1. *Let (M, g) be a spacetime satisfying TCC. Assume that M admits a globally defined conformal causal vector field K such that at any point either $\overline{\nabla}\varrho$ is a future causal vector field or it vanishes. Then, every maximal parabolic hypersurface S immersed in M must be totally geodesic. Moreover, the vector field $\overline{\nabla}\varrho$ vanishes identically on the maximal hypersurface.*

Proof. From the assumptions of the theorem and (5) it immediately follows that $\Delta\Phi \leq 0$. Since S is parabolic Φ should be constant, and consequently $\Delta\Phi = 0$. Then all the terms in the right hand side of (5) should also vanish, and in particular $\operatorname{tr}(A^2) = 0$ and $\overline{\nabla}\varrho = 0$ on S. □

Remark 3.1. Some observations are in order:

(i) Note that Theorem 3.1 can be slightly generalized to the case of constant mean curvature spacelike hypersurfaces whose mean curvature H satisfies $\varrho H \geq 0$ on the hypersurface. Beside, let us observe that the assumption on $\overline{\nabla}\varrho$ is trivially satisfied in the homothetic case, i.e., when the conformal factor is constant.

(ii) Previous result follows in particular if we assume that (M, g) is either a conformally stationary spacetime or a pp-wave. Observe that, in particular, for stationary spacetimes or pp-waves (where K is actually a Killing vector), the conformal factor ϱ vanishes.

(iii) Regarding the case of pp-waves, it is worth pointing out that it is easy to construct non-trivial pp-waves which admits parabolic spacelike hypersurfaces as it is shown in the following example. Let (M, g) be a Lorentzian $(n+1)$-dimensional manifold where M is given by the product $\Sigma \times \mathbb{S}^1 \times \mathbb{R}$, endowed with the Lorentzian metric

$$g = g_{\Sigma} + 2d\alpha dv + H(x, \alpha)d\alpha^2,$$

(Σ, g_{Σ}) being a (complete) parabolic Riemannian $(n-1)$-manifold, $\mathbb{S}^1$ the unitary sphere, α an angular coordinate on $\mathbb{S}^1$ and $v \in \mathbb{R}$. Let us note that the vector field $\frac{\partial}{\partial v}$ is a global parallel lightlike vector field on M, and taking into account that the Riemannian product of a parabolic with a compact Riemannian manifold is parabolic, we obtain that M admits a parabolic spacelike hypersurface, as long as H is positive.

(iv) Finally, notice that following the Brinkmann program and given a pp-wave spacetime (M, g), we can obtain a spacetime with a non-necessarily parallel conformal lightlike vector field. Indeed, it is enough to consider the family of spacetimes $(M, \tilde{g})$, where the metric $\tilde{g}$ is conformal to the original metric g. Furthermore, under a suitable choice of the conformal metric $\tilde{g}$, every parabolic spacelike hypersurface in (M, g) is again a parabolic spacelike hypersurface in $(M, \tilde{g})$ (see [14] for details). As a direct consequence, there is a great family of spacetimes admitting a lightlike conformal vector field where our result holds.

Theorem 3.1 has a bunch of nice consequences. On the one hand, and from the proof of Theorem 3.1, it follows that under the assumption of TCC, $\overline{\mathrm{Ric}}(N, N) = 0$ over any maximal parabolic hypersurface. Hence, we can obtain non-existence results by hardening the TCC condition. In this sense, let us remind that a spacetime is said to satisfy the ubiquitous convergence condition (UCC) if $\overline{\mathrm{Ric}}(Z, Z) > 0$ for any timelike vector field Z. UCC is a stronger energy condition than TCC which roughly means a real presence of matter at any event of the spacetime. Replacing the timelike convergence condition by UCC in the previous result, we get:

Corollary 3.2. *Let (M, g) be a spacetime satisfying UCC and assume it admits a globally defined conformal causal vector field K such that at any point either $\overline{\nabla}\varrho$ is a future causal vector field or it vanishes. Then there does not exist any maximal parabolic hypersurface in M. In particular, there do not exist maximal parabolic hypersurfaces immersed in a stationary spacetime or a pp-wave obeying UCC.*

On the other hand, the result is directly applicable for the case of gravitational waves, a particular subfamily of pp-waves obtained when we ask them to be a vacuum solution (i.e. Ricci flat). Recall that these spacetimes are physically represent purely gravitational radiation propagating along certain null rays (for details, see [3, Chapter 13]).

Corollary 3.3. *Let (M, g) be a gravitational plane wawe spacetime, then every constant mean curvature parabolic spacelike hypersurface in M must be totally geodesic.*

Finally, when considering complete, *non-parabolic* spacelike hypersurfaces, we need the following technical result, obtained by Caminha, of a well-kwown result by Yau [16]. Given M a Riemannian manifold, let $\mathcal{L}^1(M)$ be the space of Lebesgue integrable functions on M and div_M the divergence operator on M, then [6, Proposition 2.1] states that

Proposition 3.4. *Let X be a smooth vector field on a complete, noncompact, oriented Riemannian manifold Σ, such that $\mathrm{div}_\Sigma(X)$ does not change sign on Σ. If $|X| \in \mathcal{L}^1(\Sigma)$, then $\mathrm{div}_\Sigma(X) = 0$ on Σ.*

As a consequence of Proposition 3.4 we can get the following theorem,

Theorem 3.2. *Let (M, g) be a spacetime satisfying TCC, which admits a globally defined conformal causal vector field K such that at any point $\overline{\nabla}\varrho$ is a future causal vector field or it vanishes and let S be a complete spacelike hypersurface in M with constant mean curvature such that $|\nabla\Phi + HK^\top| \in \mathcal{L}^1(S)$. Then S is totally umbilical.*

Proof. From the fact of K being conformal and taking into account that $K^\top = K + g(K, N)N$, it easily follows that

$$\mathrm{div}(K^\top) = n\varrho + \Phi n H,$$

which jointly with (5) yields

$$\mathrm{div}(\nabla\Phi + HK^\top) = -\Phi(\mathrm{tr}(A^2) - nH^2 + \overline{\mathrm{Ric}}(N, N)) + ng(N, \overline{\nabla}\varrho). \tag{6}$$

From the assumptions of the theorem and Proposition 3.4 $\mathrm{div}(\nabla\Phi + HK^\top) = 0$, so all the terms in the right hand side in (6) should also vanish. In particular $\mathrm{tr}(A^2) - nH^2 = 0$. Thus, S is a totally umbilical hypersurface. □

3.1. Spacetimes admitting a closed conformal causal vector field

As a final subsection of this note, let us sharpen Theorem 3.1 by assuming geometrical conditions which ensures that $\overline{\nabla}\varrho$ is a future causal vector field. For this, let us assume that (M, g) admits a conformal vector field K which is closed, i.e., its metrically equivalent 1-form $K^\flat$ is closed. Observe that, when this conformal vector field is timelike, (M, g) belongs to an important subfamily of the class of conformally stationary spacetimes since, among other properties, they can be foliated by spacelike hypersurfaces of constant mean curvature. Moreover, since the the conformal vector field is (at least) locally a gradient, there exists for all point $p \in M$ an open neighbourhood endowed with a time function.

When K is a closed conformal causal vector field, the conformal factor ϱ associated with the conformal vector field is such that

$$\overline{\nabla}_V K = \varrho V$$

for all $V \in \mathfrak{X}(M)$. Hence, it follows that

$$d\,\|K\|^2 = 2\varrho\, K^\flat\,.$$

In particular, the 1-form $\varrho\, K^\flat$ is exact, so

$$d\varrho \wedge K^\flat = 0,$$

and thus $d\varrho = \lambda\, K^\flat$ for a certain smooth function λ. Therefore $\overline{\nabla}\varrho$ is always a causal vector field and, in order to fulfill the requirements of Theorem 3.1, we only need to ensure that it is future-directed, i.e. $\lambda \geq 0$. In this sense, observe that if we consider now V a spacelike vector field, it follows that‡

$$\begin{aligned} g\left(\overline{\mathrm{R}}_{KV}K, V\right) &= \varrho\, g\left(V, [K, V]\right) - \left(g\left(\overline{\nabla}_K \varrho V, V\right) - g\left(\overline{\nabla}_V \varrho K, V\right)\right) \\ &= -\,K(\varrho)\, g(V, V) + V(\varrho)\, g(K, V) \\ &= -\,\lambda\, \left(g(K, K)\, g(V, V) - \, g(K, V)^2\right)\,. \end{aligned}$$

As K is causal and V spacelike, the plane $K_p \wedge V_p \subset T_pM$ is timelike, and so, $(g(K, K)\, g(V, V) - \, g(K, V)^2)$ is negative. Therefore, in order to ensure that $\lambda \geq 0$, we only need to assume that for each point $p \in M$ there exists a timelike plane with non-positive sectional curvature. This follows for instance if the TCC is satisfied.

Acknowdlegements

The first author is partially supported by MICINN/FEDER project PGC2018-097046-B-I00 and Fundación Séneca project 19901/GERM/15, Spain. Her work is a result of the activity developed within the framework of the Program in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Science and Technology Agency of the Región de Murcia. The second author is partially supported by the Spanish Grant MTM2016-78807-C2-2-P (MINECO and FEDER funds). The third author is partially supported by the Spanish MINECO and ERDF project Grant MTM2016-78807-C2-1-P.

References

[1] L. J. Alías, A. Romero, and M. Sánchez, *Spacelike Hypersurfaces of Constant Mean Curvature in Certain Spacetimes*, Nonlinear Analysis: Theory, Methods & Applications, 30 (1997), pp. 655–661.

[2] A. Barros, A. Brasil, and A. Caminha, *Stability of spacelike hypersurfaces in foliated spacetimes*, Differential Geometry and its Applications, 26 (2008), pp. 357–365.

‡ We adopt for the $(1,3)$-curvature tensor of the spacetime the following definition ([12, Chapter 3]): for $X, Y, Z \in \mathfrak{X}(M)$, $\overline{\mathrm{R}}_{XY}Z = \overline{\nabla}_{[X,Y]}Z - \left[\,\overline{\nabla}_X, \overline{\nabla}_Y\right] Z$; defining the Ricci tensor accordingly.

[3] J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian Geometry*, Pure and Applied Mathematics, Marcel Dekker, New York, 1996.

[4] H. W. Brinkmann, *Riemann spaces conformal to Einstein spaces*, Mathematische Annalen, 91 (1924), pp. 269–278.

[5] F. Camargo, A. Caminha, H. de Lima, and M. A. Velásquez, *On the geometry of conformally stationary Lorentz spaces*, Acta Mathematica Hungarica, 134 (2012), pp. 385–403.

[6] A. Caminha, *The geometry of closed conformal vector fields on Riemannian spaces*, Bulletin of the Brazilian Mathematical Society, 42 (2011), pp. 277–300.

[7] Y. Choquet-Bruhat, *The Problem of Constraints in General Relativity: Solution of the Lichnerowicz Equation*, in Differential Geometry and Relativity, M. Cahen and M. Flato, eds., Mathematical Physics and Applied Mathematics, Springer, Dordrecht, 1976, pp. 225–235.

[8] V. Daftadar and N. Dadhich, *Gradient conformal Killing vectors and exact solutions*, General Relativity and Gravitation, 26 (1994), pp. 859–868.

[9] D. Earley, J. Isenberg, J. Marsden, and V. Moncrief, *Homothetic and conformal symmetries of solutions to Einstein's equations*, Communications in Mathematical Physics, 106 (1986), pp. 137–158.

[10] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1973.

[11] J. D. Marsden and F. J. Tipler, *Maximal hypersurfaces and foliations of constant mean curvature in General Relativity*, Physics Reports, 66 (1980), pp. 109–139.

[12] B. O'Neill, *Semi-Riemannian Geometry With Applications to Relativity*, vol. 103 of Pure and Applied Mathematics, Academic Press, 1983.

[13] J. A. S. Pelegrín, A. Romero, and R. M. Rubio, *On maximal hypersurfaces in lorentz manifolds admitting a parallel lightlike vector field*, Classical and Quantum Gravity, 33 (2016), p. 055003.

[14] A. Romero, R. M. Rubio, and J. J. Salamanca, *A new approach for uniqueness of complete maximal hypersurfaces in spatially parabolic GRW spacetimes*, Journal of Mathematical Analysis and Applications, 419 (2014), pp. 355–372.

[15] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaersand, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 2003.

[16] S. T. Yau, *Some Function-Theoretic Properties of Complete Riemannian Manifolds and their Applications to Geometry*, Indiana University Mathematics Journal, 25 (1976), pp. 659–670.