Calabi-Bernstein type problems in Lorentzian Geometry

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Abstract

The study of maximal hypersurfaces in Lorentzian manifolds is an interesting mathematical problem, which connects differential geometry, nonlinear partial differential equations and certain problems in mathematical relativity. One of the more celebrated result in the context of global geometry of maximal hypersurfaces is the Calabi-Bernstein theorem in the Lorentz-Minkowski spacetime. The non-parametric version of this theorem states that the only entire solutions to the maximal hypersurface equation in the Lorentz-Minkowski spacetime are spacelike affine hyperplanes. The present work reviews some of the classical and recent proofs of the theorem for the two dimensional case, as well as several extensions for Lorentzian warped products and other relevant spacetimes. On the other hand the problem of uniqueness of complete maximal hypersurfaces is analysed under the perspective of some new results.

1 Introduction

We begin with two examples of nonlinear partial differential equations, which arise in the context of some differential geometric problem.

First, we recall the well-known minimal hypersurface equation in the Euclidean space $\mathbb{R}^{n+1}$. So, for a smooth function $u : \Omega \longrightarrow \mathbb{R}$ on a domain Ω in $\mathbb{R}^n$, the problem is given by the following non-linear differential equation in divergence form,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0, \tag{1}$$

where D and div denote the gradient and divergence operators in the Euclidean n-plane $\mathbb{R}^n$ respectively. This equation is elliptic and it is easy to see that the affine functions are trivial solutions.

Secondly, the maximal spacelike hypersurface equation in the Lorentz-Minkowski spacetime $\mathbb{L}^{n+1}$. With coordinates $(t, x_1, ..., x_n)$ (and Lorentzian form $g = -dt^2 + \sum_{j=1}^{n} dx_j^2$), the problem is given for $t = u(x_1, ..., x_n)$ by

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$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |\, Du \,|^2 < 1. \tag{2}$$

where D and div denote the gradient and divergence operators in the Euclidean n-plane $\mathbb{R}^n$ respectively.

The condition $|\, Du \,|^2 < 1$ assures that the graph of every solution is spacelike, this is, the induced fundamental form on the graph is definite positive. Moreover, the problem is elliptic thanks to this extra constraint.

Note that if we take an unit normal vector field on the graph $t = u(x_1, ..., x_n)$ in the same time-orientation of the timelike coordinate vector field $\frac{\partial}{\partial t} := \partial_t$, then its mean curvature is given by

$$nH = \operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right).$$

On the other hand, the graph of u is extremal, among functions satisfying the spacelike condition under interior variations (with compact support) for the volume integral,

$$V = \int \sqrt{1 - |\, Du \,|^2}\, dx_1 \wedge .. \wedge dx_n.$$

Again, trivial solutions of equation (2) are affine functions (with spacelike graph).

1.1 Bernstein theorem

The early seminal result of S. Bernstein [13], amended by E. Hopf [35], is the well-known following uniqueness theorem,

> *The only entire solutions to the equation (1) on the Euclidean plane $\mathbb{R}^2$ are the affine functions.*

This result is known as the classical Bernstein theorem. In 1968, J. Simons [61] proved a result which in combination with theorems of E. De Giorgi [32] and W.H. Fleming [30] yield a proof of the Bernstein theorem for $n \leq 7$. Moreover, there is a counterexample $u \in C^\infty(\mathbb{R}^n)$ to the Bernstein conjecture for each $n \geq 8$.

1.2 Calabi-Bernstein Theorem

One of the most relevant results in the context of global geometry of spacelike surfaces is the classical Calabi-Bernstein theorem. This result was established in 1970 by Calabi [21] inspired in the classical Bernstein theorem, via a duality between solutions to equations (1) and (2).

In its non-parametric version, it asserts that the only entire solutions to the maximal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |Du| < 1$$

on the Euclidean plane $\mathbb{R}^2$ are affine functions.

In fact, Calabi also shows that the result holds for the case of maximal hypersurfaces in $\mathbb{L}^4$. Later on, Cheng and Yau [22] extended the Calabi-Bernstein theorem to the general $n+1$-dimensional case. Another important achievement in [22] was the introduction of a new procedure, the so-called Omori-Yau generalized maximum principle [44], [63].

The Calabi-Bernstein Theorem can also be formulated in a parametric way. In this case, it states that the only complete maximal hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. In their proof of the parametric version, Cheng and Yau obtain a Simons-type formula, this is, the authors compute the Laplacian of the trace of the square of the associate shape operator to the unitary normal vector field on the maximal hypersurface. Subsequently, assuming completeness and making use of a consequence of their new maximum principle, the authors obtain the result in parametric version. Nevertheless, both versions (parametric and non-parametric ones) are not equivalent a priori, since there exist examples of spacelike entire graphs in $\mathbb{L}^{n+1}$ which are not complete (see for instance [6]). This fact, is a notable difference and difficulty with respect to the Riemannian case, where thank to the Hopf-Rinow theorem all entire graph in $\mathbb{R}^{n+1}$ must be complete. So, Cheng and Yau prove that in the case where the mean curvature is constant, a embedded spacelike hypersurface in $\mathbb{L}^{n+1}$ must be complete, which allows to obtain the non-parametric version.

2 Some approaches to the classical Calabi-Bernstein theorem

After the general proof by Cheng and Yau, several authors have approached to the classical version of Calabi-Bernstein theorem from different perspectives, providing diverse extensions and new proofs of the result in $\mathbb{L}^3$.

In 1983, Kobayashi [38] derived the Calabi-Bernstein Theorem as a consequence of the corresponding Weierstrass-Enneper parameterization for maximal surfaces in $\mathbb{L}^3$. Below we briefly describe the proof of Kobayashi.

2.1 Kobayashi approach

Consider the Lorentz-Minkowski spacetime $\mathbb{L}^3$, which is given by the Lorentzian manifold $(\mathbb{R}^3, \langle\, ,\rangle)$, where

$$\langle\, ,\rangle = -dt^2 + dx^2 + dy^2.$$

Let S be a connected maximal surface in $\mathbb{L}^3$. The surface S must be orientable and let N be the unitary normal vector field on S such that $\langle N, \frac{\partial}{\partial t}\rangle < 0$. Let $N : S \longrightarrow \mathbb{H}^2_+$ be the Gauss map of S and define a sterereographic projection $\sigma : \mathbb{D} \longrightarrow \mathbb{H}^2_+$, from $\mathbb{D} = \{z \in \mathbb{C} / \mid z \mid < 1\}$ onto $\mathbb{H}^2_+$ as follow

$$\sigma(z) = \Big(\frac{2\,\mathrm{Re}(z)}{1- \mid z \mid^2}, \frac{2\,\mathrm{Im}(z)}{1- \mid z \mid^2}, \frac{1+ \mid z \mid^2}{1- \mid z \mid^2}\Big). \tag{3}$$

The map σ is conformal and bijective assigning to each point $z \in \mathbb{D}$, the point in $\mathbb{H}^2_+$ obtained as the intersection of the straight line deternmined by $(0, 0-1)$ and $(z, 0)$ with $\mathbb{H}^2_+$.

On the other hand, the Gauss map is also conformal. Taking this into account the author shows the following result (Weierstrass-Enneper formula),

Theorem 2.1 *Any maximal surface S in $\mathbb{L}^3$ is represented as*

$$\phi(z) = \mathrm{Re} \int \Big(\frac{1}{2} f(1+g^2), \frac{i}{2} f(1-g^2), -fg\Big) dz, \;\; z \in D, \tag{4}$$

where D is a domain in $\mathbb{C}$, and f (resp. g) is a holomorphic (resp. meromorphic) function on D such that fg^2 is holomorphic in D and $\mid g(z) \mid \neq 1$ for $z \in D$. Moreover,

(i) The Gauss map N is given by $N(z) = \sigma(g(z))$, where σ is the map defined in (3).
(ii) The induced metric is given by $ds = \frac{1}{2} \mid f \mid (\mid 1- \mid g \mid^2 \mid) \mid dz \mid$.
(iii) The Gauss curvature of the surface is given by $K = \big(\frac{4 |\partial_z g|}{|f|(|1-|g|^2|)^2}\big)^2$.

Thus, we consider the inmersion $\phi : D \longrightarrow \mathbb{L}^3$ with the induced metric from $\mathbb{L}^3$.

Assume that the maximal surface is complete. Since the surface is not compact, the uniformization theorem allow us to affirm that D must be conformal to $\mathbb{C}$ or the unit disc $\mathbb{D}$. Suppose that D is conformal to $\mathbb{D}$. Now, the author makes use of a result of Osserman [46, p. 67] to show, that there is a divergent curve γ in D such that $\int_\gamma \mid f(z) \mid \mid dz \mid < \infty$, thus using (ii) of Theorem (2.1) we will conclude that the surface cannot be complete. Hence, D must be conformal to $\mathbb{C}$. Finally, taking into account (i) in Theorem (2.1) we have $\mid g \mid < 1$ and g is holomorphic. Now, it is enough to call Liouville's theorem to obtain that g is a constant function and as a direct consequence the immersion is a spacelike plane.

2.2 On other approaches

In 1994, a new proof of the Classical Calabi-Bernstein theorem is given by Estudillo and Romero [29] making use again of the Weierstrass-Enneper representation. The authors find an adequate local upper bound for the Gaussian curvature of a maximal surface. Estudillo and Romero inspired in a paper by Osserman [47] obtain the following inequality of the Gauss curvature at any point p, $K(p)$, of a maximal surface S with boundary in $\mathbb{L}^3$,

$$K(p) \leq \frac{4}{d(p, \partial S)^2}, \quad \text{for any} \quad p \in S,$$

where d is the Riemannian distance on S.

As a consequence, if we consider a complete maximal surface S and an arbitrary point $p \in S$, we can take a geodesic disc with center at p and radius r. Now, it is enough to choose r as large as we desire to conclude that S must be totally geodesic.

On the other hand, using a conformal metric, the authors get a new proof of the non-parametric version.

In the real field, a simple proof of the non-parametric version, which only requires the Liouville theorem for harmonic functions on the Euclidean plane $\mathbb{R}^2$ was given in 1994 by Romero [52]. As the author says, the proof is inspired in a proof of the classical Bernstein theorem given by Chern [26]. So, the author obtains a conformal metric on the entire graph, which is complete and flat. Thus, via Cartan's theorem, the graph endows with the conformal metric is isometric to Euclidean plane. On the other hand, Romero shows that the function $\frac{1}{\langle N, a \rangle}$, where N is the unitary (future directed) normal vector field on the graph and a is an suitable constant lightlike vector, is a positive harmonic function globally defined on the graph.

Finally, taking into account, the invariance of harmonic functions by conformal changes of metric we have that $\langle N, a\rangle$ is constant.

Via a local integral inequality for the Gaussian curvature of a maximal surface, in 2000, Alías and Palmer [7] provided another new proof for the parametric case. The authors obtain an upper bound for the total curvature of geodesic discs in a maximal surface in terms of the local geometry of the surface and its hyperbolic image. Specifically, the authors show

Theorem 2.2 *Let $x : S \longrightarrow \mathbb{L}^3$ be a maximal surface in the three-dimensional Lorentz-Minkowski spacetime. Let $p \in S$ and $R > 0$ be a positive real number such that the geodesic disc of radius R about p satisfies $D(p, R) \subset\subset S$. Then for all $0 < r < R$, the total curvature of a geodesic disc $D(p, r)$ satisfies*

$$0 \leq \int_{D(p,r)} K dA \leq c_r \frac{L(r)}{r \log(R/r)}, \tag{5}$$

where dA is the area element, $L(r)$ denotes the lenght of $\partial D(p, r)$ and $c_r = c_r(p, r)$ is a contant.

Making use of this integral inequality, Alías and Palmer get a new proof of the Calabi-Bernstein theorem. Indeed, if S is complete, R can approach to infinity for any fixed point p and fixed radius r, now from (5) $K \equiv 0$.

These authors get also a new proof of the non-parametric version based on a duality result with minimal surface equation in the Euclidean case [9]. Recently (2010), yet another short proof of both versions has been given by Romero and Rubio [53] making use of the interface between the parabolicity of a Riemannian surface and the capacity of geodesic annuli. Finally, a more recent (2015) original new proof has been given by Aledo, Romero and Rubio by using a development inspired by the well-known Bochner's technique.

We must emphasize that for several of the proof of the classical result, it is essential in a way or another that any complete maximal surface in $\mathbb{L}^3$ must be parabolic.

2.3 Romero-Rubio's proof of the classical result

In this section we will describe the new approach given by Romero and Rubio [53] to the two-dimensional version of the Calabi-Bernstein theorem.

Consider the Lorentz-Minkowski spacetime $\mathbb{L}^3$, which is given by the Lorentzian manifold $(\mathbb{R}^3, \langle\, , \rangle)$, where

$$\langle\, , \rangle = -dt^2 + dx^2 + dy^2.$$

Let $x : S \longrightarrow \mathbb{L}^3$ be a (connected) immersed spacelike surface in $\mathbb{L}^3$. Observe that S must be orientable and let N be the unitary normal vector field on S such that $\langle N, \partial_t\rangle > 0$, where ∂_t denotes the coordinate vector field $\frac{\partial}{\partial t}$. If $\theta(p)$ denotes the hyperbolic angle between N and $-\partial_t$ at $p \in S$, then $\cosh\theta = \langle N, \partial_t\rangle$.

We will denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{L}^3$ and S, respectively. Then the Gauss and Weingarten formulas for S in $\mathbb{L}^3$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle A(X), Y\rangle N \tag{6}$$

and

$$A(X) = -\overline{\nabla}_X N, \tag{7}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(S)$, where $A : \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$ stands for the *shape operator* associated to N.

On the other hand, the tangential component of ∂_t at any point of S is given by $\partial_t^\top = \partial_t + \cosh\theta N$.

We suppose that S is maximal. It is immediate to see that

$$\nabla \cosh\theta = -A\partial_t^\top$$

being A the shape operator associated to N. It is not difficult to obtain by standard computation the following formulas:

$$\mid \nabla \cosh\theta \mid^2 = \frac{1}{2}\text{trace}\,(A^2)\sinh^2\theta \quad \text{and} \quad \Delta\cosh\theta = \text{trace}\,(A^2)\cosh\theta$$

where ∇ and Δ are respectively the gradient and Laplacian relative to the induced Riemannian metric on S.

The following technical result, which is a reformulation by Romero and Rubio [53] of a Lemma by Alías and Palmer [8], is a key piece of the this new approach. Previously, we need to recall some general preliminaries.

Let S be an $n(\geq 2)$-dimensional Riemannian manifold and let B_r denote the geodesic ball of radius r around a fixed point $p \in S$. For $0 < r < R$ let $A_{r,R}$ be the geodesic annulus $A_{r,R} := B_R \setminus \bar{B}_r$. Denote by $\omega_{r,R}$ the harmonic measure of ∂B_R with respect to $A_{r,R}$, that is the solution of the elliptic problem

$$\Delta\omega = 0 \quad \text{in} \quad A_{r,R}, \quad \omega \equiv 0 \quad \text{on} \quad \partial B_r, \quad \text{and} \quad \omega \equiv 1 \quad \text{on} \quad \partial B_R.$$

The capacity of the annulus is defined to be

$$\frac{1}{\mu_{r,R}} := \int_{A_{r,R}} |\nabla\omega_{r,R}|^2 \, dV.$$

It is well known that S is parabolic if and only if

$$\lim_{R\to\infty} \frac{1}{\mu_{r,R}} = 0.$$

Now, we can enunciate the technical lemma.

Lemma 2.3 *Let S be an $n(\geq 2)$-dimensional Riemannian manifold and let $v \in C^2(S)$ which satisfies $v\Delta v \geq 0$. Let B_R be a geodesic ball of radius R in S. For any r such that $0 < r < R$ we have*

$$\int_{B_r} |\nabla v|^2 \, dV \leq \frac{4\,\text{Sup}_{B_R} v^2}{\mu_{r,R}},$$

where B_r denote the geodesic ball of radius r around p in S and $\frac{1}{\mu_{r,R}}$ is the capacity of the annulus $B_R \setminus \bar{B}_r$.

Proof. For any $\zeta \in C^\infty(B_R)$ with $\mathrm{supp}(\zeta) \subset B_R$, from the divergence theorem we have

$$\int_{B_R} (\zeta^2|\nabla v|^2 + \zeta^2 v\triangle v + 2\zeta v\langle\nabla\zeta,\, \nabla v\rangle)dV = 0,$$

and as a consequence,

$$\int_{B_R} \zeta^2|\nabla v|^2 dV \leq 2\int_{B_R} |\zeta v\langle\nabla\zeta,\, \nabla v\rangle| dV \leq a^2\int_{B_R} \zeta^2|\nabla v|^2 dV + \frac{1}{a^2}\int_{B_R} v^2|\nabla\zeta|^2 dV,$$

for all $a > 0$, and hence taking $a = 1/\sqrt{2}$ we obtain

$$\int_{B_R} \zeta^2|\nabla v|^2 dV \leq 4\int_{B_R} v^2|\nabla\zeta|^2 dV \leq 4\,\mathrm{Sup}_{B_R} v^2\int_{B_R} |\nabla\zeta|^2 dV.$$

Define ζ by

$$\zeta(x) = \begin{cases} 1 & \text{if } x \in \bar{B}_r \\ 1 - \omega_{r,R} & \text{if } x \in A_{r,R} \end{cases}$$

Finally, although ζ is not smooth it can be approximated by smooth function, and so we obtain

$$\int_{B_r} |\nabla v|^2 dV \leq \frac{4\,\mathrm{Sup}_{B_R} u^2}{\mu_{r,R}}.$$

□

We are now in a position to describe the proof of the parametric version. So, consider the auxiliary fuction $v : S \longrightarrow (\frac{\pi}{2}, \frac{3\pi}{2})$, $v(p) = \arctan(\cosh\theta(p))$, which has an advantage on the original $\cosh\theta$, this is, v is bounded.

It is immediate to verify that $v\Delta v \geq 0$. From the previous Lemma, and taking into account that

$$\nabla v = \frac{1}{1 + \cosh^2\theta}\nabla\cosh\theta,$$

we have

$$\int_{B_r} |\nabla v|^2\, dV \leq \frac{9\pi^2}{\mu_{r,R}},$$

for $0 < r < R$, which easily gives

$$\int_{B_r} |\nabla(\cosh\theta)|^2\, dV \leq \frac{C}{\mu_{r,R}},$$

where B_r denote the geodesic disc of radius r around p in S, $\frac{1}{\mu_{r,R}}$ is the capacity of annulus $B_R \setminus \bar{B}_r$ and $C = C(p, r) > 0$ is constant.

Now, the surface S is necessarily non compact and from the Gauss formula it has curvature $K \geq 0$. If we assume that S is complete, a classical result by Ahlfors and Blanc-Fiala-Huber (see for instance, [37]), affirms that a complete 2-dimensional Riemannian manifold with non-negative Gauss curvature is parabolic.

On other hand, it is well know that S will be parabolic if and only if $\lim_{R\to\infty}\frac{1}{\mu_{r,R}} = 0$. We get that R can approach to infinity for a fixed arbitrary point p and a fixed r, obtaining that $\cosh\theta$ is constant on S.

2.4 The non-parametric case

We finish the Romero and Rubio approach with a a sketch of the proof given by the authors for the non-parametric version of the classical Calabi-Bernstein theorem.

For each $u \in C^\infty(\Omega)$, note that the induced metric on $\Omega \subset \mathbb{R}^2$, via the graph $\big\{(u(x,y),x,y) : (x,y) \in \Omega\big\} \subset \mathbb{L}^3$, is $g_u := -du^2 + g_{_0}$, where $g_{_0}$ is the usual Riemannian metric of $\mathbb{R}^2$. The metric g_u is positive definite, if and only if u satisfies $|Du| < 1$, where Du denote the gradient of u in $(\mathbb{R}^2, g_0)$.

On the other hand, the graph of u is spacelike and has zero mean curvature if and only if u is a solution to the maximal surface equation (2) in the Lorentz-Minkowski space.

We consider on $\mathbb{R}^2$ the function $\cosh\theta = \frac{1}{\sqrt{1-|Du|^2}}$ and the conformal metric $g' = (\cosh\theta + 1)^2 g_u$, which taking into account the relation between curvatures for conformal changes (see for instance, [14]) is flat.

If the graph is entire, then g' is complete, because $L' \geq L_0$ where L' and L_0 denote the lengths of a curve on $\mathbb{R}^2$ with respect to g' and the usual metric of $\mathbb{R}^2$. Taking into account the invariance of subharmonic functions by conformal changes of metric, we are in position to use the same argument as in the parametric case on the Riemannian surface (F, g') to get the result.

2.5 A new proof using the Bochner technique

Recently, yet another proof of the classical Calabi-Bernstein theorem have been given by Aledo Romero and Rubio [4]. In this paper the authors make use of the Bochner technique. By mean of the Bochner-Lichnerowicz's Formula and a well-known Liouville type result, the authors show the parametric version of the aforementioned theorem.

Next, we will explain the main steps of the Aledo-Romero-Rubio's proof.

Consider $x : S \longrightarrow \mathbb{L}^3$ a (connected) immersed maximal surface in the Lorentz-Minkowski spacetime $\mathbb{L}^3$. We choose a unit timelike normal vector field N globally defined on S in the same time-orientation of $\frac{\partial}{\partial t}$.

Making use of the Gauss equation for a surface in $\mathbb{L}^3$, it is easy to verify that

$$\mathrm{trace}(A^2) = 2K, \tag{8}$$

where A denotes the shape operator associated to the normal vector field N and K is the Gaussian curvature of the surface.

The idea of the proof is to choose a suitable function on the maximal surface and to apply the Bochner-Lichnerowicz's Formula.

Recall that the well-known Bochner-Lichnerowicz's Formula (see, for instance [23]) states that

$$\frac{1}{2}\triangle\left(|\nabla u|^2\right) = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla(\triangle u)\rangle \tag{9}$$

for $u \in \mathcal{C}^\infty(S)$. Here Ric stands for the Ricci tensor of S and $|\mathrm{Hess}\, u|^2$ is the square algebraic trace-norm of the Hessian of u, namely $|\mathrm{Hess}\, u|^2 := \mathrm{trace}(H_u \circ H_u)$ where H_u denotes the operator defined by $\langle H_u(X), Y\rangle := \mathrm{Hess}\,(u)(X,Y)$ for all $X, Y \in \mathfrak{X}(S)$.

Let us choose $a \in \mathbb{L}^3$ a null vector, i.e., a non-zero vector such that $\langle a, a\rangle = 0$, and consider the function $\langle N, a\rangle$ on S.

Now, applying Schwarz's inequality (for symmetric square matrix), we have,

$$|\text{Hess}\,\langle N, a\rangle|^2 \geq \frac{1}{2}(\triangle\langle N, a\rangle)^2. \tag{10}$$

On the other hand, from the Weingarten formula (7) it is easy to obtain the gradient of the function $\langle N, a\rangle$ on S,

$$\nabla\langle N, a\rangle = -A(a^\top), \tag{11}$$

where $a^\top = a + \langle N, a\rangle N$ is tangent to S and standard computations allow us to obtain

$$\triangle\langle N, a\rangle = \langle N, a\rangle \text{trace}(A^2). \tag{12}$$

From (11) and taking into account that $|a^\top|^2 = \langle N, a\rangle^2$ and that S is maximal, we get

$$|\nabla\langle N, a\rangle|^2 = K\langle N, a\rangle^2 \tag{13}$$

and so

$$\text{Ric}(\nabla\langle N, a\rangle, \nabla\langle N, a\rangle) = K|\nabla\langle N, a\rangle|^2 = K^2\langle N, a\rangle^2. \tag{14}$$

With the previous computations, we can to apply the Bochner-Lichnerowicz's Formula to the choosed function $\langle N, a\rangle$ on S and so, to obain the following inequality

$$\triangle K \geq 4K^2. \tag{15}$$

Since the Gauss curvature of S is non-negative and if we assume that S is complete, then we can use the following known result (see, for instance [62]),

Lemma 2.4 *Let S be a complete Riemannian surface whose Gaussian curvature is bounded from below and $u \in \mathcal{C}^\infty(S)$ a non-negative function such that $\triangle u \geq cu^2$ for a positive constant c. Then u vanishes identically on S.*

As a consequence, $K \equiv 0$ and so S is totally geodesic.

Observe that totally geodesic spacelike surfaces in Minkowski spacetime $\mathbb{L}^3$ are spacelike planes. Nevertheless, in Lorentzian warped products this is not necessarily true, and this is why some additional hypotheses are sometimes needed in theses spaces.

3 Some extension of the classical result

In this section, we will describe some recent extensions of the Calabi-Bernstein theorem in the two-dimensional case, as well as, others Calabi-Bernstein type problems. The three-dimensional Lorentzian ambient will be given by a Lorentzian product or in a more general case for a Lorentzian warped product.

Consider (F, g) a Riemannian manifold, let $(I, -dt^2)$ be a real interval with negative metric, and $f : I \longrightarrow \mathbb{R}$ a smooth positive function. Recall that the warped product $I \times_f F$ is given by the Lorentzian manifold $(M = I \times F, \langle\, ,\rangle)$, where

$$\langle\, ,\rangle = -\pi_I^*(dt^2) + f(\pi_I)^2\pi_F^*(g), \tag{16}$$

and π_I, π_F denote the projections from M onto the base I and the fiber F, respectively.

In particular, when $f \equiv 1$ we have the Lorentzian product of $(I, -dt^2)$ and (F, g).

Recall that any warped product $I \times_f F$ possesses an infinitesimal timelike conformal symmetry which is an important tool. Indeed, the vector field

$$\xi := f(\pi_I)\,\partial_t, \tag{17}$$

which is timelike and, from the relationship between the Levi-Civita connections of M and those of the base and the fiber, satisfies

$$\overline{\nabla}_X \xi = f'(\pi_I)\,X \tag{18}$$

for any $X \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the Levi-Civita connection of the warped metric. Thus, ξ is conformal with $\mathcal{L}_\xi \langle\,,\,\rangle = 2\,f'(\pi_I)\,\langle\,,\,\rangle$ and its metrically equivalent 1-form is closed.

Spacetimes given as a Lorentzian warped product $I \times_f F$ are introduced in General Relativity literature in [10] and they are called generalized Robert-Walker spacetimes (GRW).

In any GRW spacetime $M = I \times_f F$, the level hypersurfaces of the function $\pi_I : M \longrightarrow I$ constitute a distinguished family of spacelike hypersurfaces: the so-called *spacelike slices*. Along this work, we will represent by $t = t_0$ the spacelike slice $\{t_0\} \times F$. For a given spacelike hypersurface $x : S \longrightarrow M$, we have that $x(S)$ is contained in $t = t_0$ if and only if $\pi_I \circ x = t_0$ on S. We will say that S is a spacelike slice if $x(S)$ equals to $t = t_0$, for some $t_0 \in I$, and that S is contained between two slices if there exist $t_1, t_2 \in I$, $t_1 < t_2$, such that $x(S) \subset [t_1, t_2] \times F$.

If we take the unitary normal vector field to every spacelike slice given by $-\partial_t$, then the shape operator and the mean curvature of the spacelike slice $t = t_0$ are respectively $A = f'(t_0)/f(t_0)\,I$, where I denotes the identity transformation, and the constant $H = -f'(t_0)/f(t_0)$. Thus, a spacelike slice $t = t_0$ is maximal if and only if $f'(t_0) = 0$ (and hence, totally geodesic).

The following nonlinear elliptic differential equation, in divergence form represents the maximal surface equation in a three-dimensional Lorentzian warped product $I \times_f F$ ($\dim F = 2$, in this case),

$$\operatorname{div}\Big(\frac{Du}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\Big) = -\frac{f'(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}\Big(2 + \frac{\mid Du \mid^2}{f(u)^2}\Big) \tag{E.1}$$

$$\mid Du \mid < f(u) \tag{E.2}$$

where f is the warping function defined on the open interval I of the real line $\mathbb{R}$, the unknown u is a function defined on a domain Ω of the Riemannian surface (F, g), $u(\Omega) \subseteq I$, D and div denote the gradient and the divergence of (F, g) and $\mid Du \mid^2 := g(Du, Du)$.

The constraint (E.2) assures the spatiality of the graph $\{(u(p), p)/p \in \Omega\}$ and it is the ellipticity condition. From now on, we will refer to the nonlinear problem E.1+E.2 as equation (E):

On the other hand, the solutions of (E) are the extremals under interior variations for the functional

$$u \longmapsto \int f(u)\sqrt{f(u)^2 - \mid Du \mid^2}\, dA,$$

where dA is the area element for the Riemannian metric g, which acts on functions u such that $u(\Omega) \subseteq I$ and $\mid Du \mid < f(u)$.

Observe that when $I = \mathbb{R}$, $F = \mathbb{R}^2$ and $f = 1$, the equation (E) is the maximal surface equation in $\mathbb{L}^3$.

Note that a constant function $u = c$ is a solution to the equation (E), if and only if $f'(c) = 0$.

We will begin with a new example of non-parametric Calabi-Bernstein type problem given by Latorre and Romero [39]. We have to say that this paper is the first one dealing with the maximal surface equation for warped Lorentzian products, whose fiber is a complete (non-compact) 2-Riemannian manifold, in particular, the Euclidean plane $\mathbb{R}^2$. In this work, the authors assume that the sectional curvature of the Lorentzian manifold is not zero on any proper open subset, i.e., the warping function is not locally constant, although the curvature of the ambient satisfies a natural geometric assumption arising from Relativity theory, the *null convergence condition* (NCC), which says that the Ricci quadratic form on null tangent vectors is non-negative. Obviously the Calabi-Bernstein theorem is not included in this case.

In their proof, Latorre and Romero introduce a new conformal metric on the graph. With this metric, the authors prove that the entire maximal graph is complete and parabolic, which allows to conclude that the warping function restricted to the graph is superharmonic and consequently constant. Since, the warping function is not locally constant, the only entire solutions are given by the constant functions $u = c$ with $f'(c) = 0$.

Another approach to the previous problem have been obtained by Romero and Rubio [54]. Following the ideas in Alías-Palmer's papers [7], [8], the authors obtain a local integral estimate of the squared length for the gradient of a distinguished function on the maximal surface, which is constant if and only if the surface is contained in a spacelike slice $t = t_0$, with $f'(t_0) = 0$ in the warped product.

A new extension of non-parametric Calabi-Bernstein theorem in the case of a Lorentzian product $\mathbb{R} \times F$, where F denotes a Riemannian 2-manifold, with non-negative curvature, has been given by Albujer and Alías, [2], [3]. Moreover, the authors find examples of complete and non-trivial entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ denotes the hyperbolic plane [2, Example, 5.2], (see also [1] and [31]), which show the need for certain curvature assumption on the fiber for possible extensions of the Calabi-Bernstein theorem.

Recently, another Calabi-Bernstein type results in the more general ambient of a warped Lorentzian product are given by Caballero, Romero and Rubio [18], [19]. So, the authors obtain several extensions of the classical Calabi-Bernstein theorem to tree-dimensional warped products satisfying suitable energy conditions and whose fiber can be non-necessarily of non-negative Gaussian curvature in some cases. Moreover, in the particular case where the warping function is constant, the authors recover the non-parametric extension of the classical result in Lorentzian product spaces given by Albujer and Alíias.

In a different direction, yet another extension of the classical result has been given by Pelegrín, Romero and Rubio. This time, the ambient is a three-dimensional spacetime which admits a parallel lightlike vector field and obeying the energy condition known as *timelike energy condition* (TCC) [48]. Note that, it is normally argued that TCC is the mathematical translation that gravity, on average, attracts. More precisely, the authors show that in a three-dimensional Lorentzian manifold, which admits a parallel global lightlike vector field and obeys the timelike energy condition, then every complete isometrically immersed maximal surface must be totally geodesic.

Finally, we will describe with more detail a new extension of the classical Calabi-Bernstein theorem by Rubio-Salamanca [59]. In this last work, the authors study entire solutions to the maximal surface equation in a Lorentzian three-dimensional warped product, whose fiber is given by a Riemannian surface with finite total curvature.

Recall that a complete Riemannian surface has finite total curvature if the integral of the absolute value of its Gaussian curvature is finite. Of course, the Euclidean plane has finite total curvature, but note that any complete surface, whose curvature is non-negative outside a compact subset has finite total curvature. Also, it is well-known that a complete Riemannian surface has finite total curvature if the negative part of its Gaussian curvature is integrable (see, for instance [40, Section 10]).

On the other hand, examples of complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature are known [34]. Examples in a different ambient space can be found in [50].

3.1 On the proof of Rubio-Salamanca's extension

The authors deal with the maximal surface equation for Lorentzian warped product (E), when the fiber (F, g) is a complete (non-compact) Riemannian surface with finite total curvature.

In their work, the authors are mainly interested in uniqueness and non-existence results for entire solutions (i.e. defined on all F) of equation (E).

Let Ω be a domain of the Riemannian surface F, for each $u \in C^\infty(\Omega)$, $u(\Omega) \subseteq I$, the induced metric on Ω from the Lorentzian metric (16), via its graph $\Sigma_u = \{(u(p), p) : p \in \Omega\}$ in M, is written as follows

$$g_u = -du^2 + f(u)^2 g,$$

and it is positive definite, i.e. Riemannian, if and only if u satisfies $\mid Du \mid < f(u)$ everywhere on Ω.

When g_u is Riemannian, $f(u)\sqrt{f(u)^2 - \mid Du \mid^2}\, dA$ is the area element of (Ω, g_u). Therefore (E.1) of (E) is the Euler-Lagrange equation for the area functional, its solutions are spacelike graphs of zero mean curvature in $M = I \times_f F$, and this equation is called the maximal surface equation in M.

If we denote by N the unit normal vector field N on a spacelike graph Σ_u such that $\langle N, \partial_t \rangle \geq 1$ on Σ_u, where $\partial_t := \partial/\partial t \in \mathfrak{X}(M)$, then

$$N = \frac{-f(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}} \left(1, \frac{1}{f(u)^2} Du\right),$$

and the hyperbolic angle θ between $-\partial_t$ and N is given by

$$\langle N, \partial_t \rangle = \cosh\theta = \frac{f(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}.$$

On the other hand, the Lorentzian warped product spaces considered by the authors must satisfy certain natural energy condition, which turns out to have an expression in terms of the curvature of its fiber (F, g) and the warping function f. So, recall that a Lorentzian manifold obeys NCC if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies

$$\overline{\mathrm{Ric}}(Z, Z) \geq 0,$$

for any null vector Z, i.e. $Z \neq 0$ such that $\langle Z, Z \rangle = 0$.

Taking into account how the Ricci tensor of M is obtained from the Gaussian curvature of the fiber K^F and the warping function f (see for instance [45, Corollary 7.43]) it is easy to check that a Lorentzian warped product space $I \times_f F$ with a 2-dimensional fiber obeys NCC if and only if

$$\frac{K^F(\pi_F)}{f^2} - (\log f)'' \geq 0. \tag{19}$$

From now on, let's consider spacelike entire graphs. So, let $\Sigma_u = \{(u(p), p) : p \in F\}$ be, the graph of $u \in C^\infty(F)$ such that $u(F) \subseteq I$ in the Lorentzian warped product $M = I \times_f F$. Suppose that the graph is spacelike.

Note that $\pi_I(u(p), p) = u(p)$ for any $p \in F$, and so π_I on the graph and u can be naturally identified by the isometry between $(\Sigma_u, \langle\,,\rangle)$ and (F, g_u). Analogously, the differential operators ∇ and Δ in $(\Sigma_u, \langle\,,\rangle)$ can be identified with those ones ∇_u and Δ_u in (F, g_u).

If we denote $\partial_t^\top = \partial_t + \langle N, \partial_t\rangle N$ the tangential component of ∂_t on Σ_u, then it is not difficult to see

$$\nabla \pi_{I|_{\Sigma_u}} := \nabla u = -\partial_t^\top.$$

Now, suppose the graph maximal and consider the distinguished function

$$\langle N, \xi\rangle = f(u)\cosh\theta,$$

defined on the graph. It is immediate to see that

$$\nabla\langle N, \xi\rangle = -A\xi^\top,$$

where A denotes the shape operator of the graph. Taking a orthonormal frame consisting of the eigenvectors of the shape operator, we obtain

$$\mid \nabla\langle N, \xi\rangle \mid^2 = \frac{1}{2}\mathrm{trace}\,(A^2)\{\langle N, \xi\rangle^2 - f(\pi_I)^2\}. \tag{20}$$

Moreover, using the Gauss and Codazzi equations, as well as, the expression for the Ricci tensor of M (see for instance [45, Chapter 7]), it is a standard computation to obtain (via the isometry)

$$\begin{aligned}\Delta_u(f(u)\cosh\theta) &= \Big\{\frac{K^F}{f(u)^2} - (\log f)''(u)\Big\}|\nabla_u u|^2 f(u)\cosh\theta \\ &+ \frac{1}{2}\mathrm{trace}(A^2) f(u)\cosh\theta \end{aligned} \tag{21}$$

On the other hand, taking into account the Gauss equation and using again the expression for the Ricci tensor of M, then the Gauss curvature of a maximal graph is

$$K = \frac{f'(u)^2}{f(u)^2} + \Big\{\frac{K^F}{f(u)^2} - (\log f)''(u)\Big\} \mid \partial_t^\top \mid^2 + \frac{K^F}{f(u)^2} + \frac{1}{2}\mathrm{trace}(A^2). \tag{22}$$

As a direct consequence, from (21) we have the following alternative expression,

$$\Delta_u(f(u)\cosh\theta) = \Big\{K_u - \frac{f'(u)^2}{f(u)^2} - \frac{K^F}{f(u)^2} + \frac{1}{2}\mathrm{trace}(A^2)\Big\} f(u)\cosh\theta. \tag{23}$$

One of the fundamental tools in the work of Rubio and Salamenca is to introduce a conformal metric on the graph, which allows certain control on its Gaussian curvature. So, on the manifold F we consider the following Riemannian metric

$$g'_u := f(u)^2\cosh^2\theta\, g_u, \tag{24}$$

where

$$f(u)\cosh\theta = \frac{f(u)^2}{\sqrt{f(u)^2 - \mid Du\mid^2}}$$

and $\mid Du \mid^2 := g(Du, Du)$. Therefore, if $\epsilon := \mathrm{Inf}(f) > 0$ we get the following inequality

$$L' \geq \epsilon^2 L,$$

where L' and L denote the lengths of a curve in F with respect to g'_u and g, respectively. Consequently, g'_u is complete whenever g is complete.

Now, suppose that $\mathrm{Sup} f(u) < \infty$. Put $\lambda = \mathrm{Sup} f(u)$ and consider the new Riemannian metric

$$g^*_u := (f(u)\cosh\theta + \lambda)^2 g_u \tag{25}$$

on F. The completeness of the metric (24) assures that g^*_u is also complete. Moreover, it has the advantage over g'_u that we can control its Gaussian curvature under reasonable assumptions. In order to concrete this assertion, denote by K^*_u and K_u the Gaussian curvatures of the Riemannian metrics g^*_u and g_u, respectively. From (25) and using the relation between Gaussian curvatures for conformal changes (see for instance, [13]) , we have

$$K_u - (f(u)\cosh\theta + \lambda)^2 K^*_u = \Delta_u \log(f(u)\cosh\theta + \lambda). \tag{26}$$

The following lemma is key to the achievement of the principal result, since it allows to assure that the graph endowed with the appropriate conformal metric will has finite total curvature.

Lemma 3.1 *Suppose that (F, g) is complete, with finite total curvature. If $\mathrm{Inf} f > 0$, $\mathrm{Sup} f < \infty$ and the inequality $\frac{K^F}{f(u)^2} - (\log f)''(u) \geq 0$ holds on F, then the complete Riemannian surface (F, g^*_u) has finite total curvature.*

Proof. From the previous expressions (22) and (21) we get,

$$\Delta_u \log(f(u)\cosh\theta + \lambda) \leq \frac{1}{f(u)\cosh\theta + \lambda}\left\{\Big(K_u - \frac{K^F}{f(u)^2}\Big) f(u)\cosh\theta + \Big(K_u - \frac{K^F}{f(u)^2}\Big)\lambda\right\}$$

$$\leq K_u - \frac{K^F}{f(u)^2}.$$

Since the Riemannian area elements of the metrics g and g^*_u satisfy

$$dA^*_u = \frac{(f(u)\cosh\theta + \lambda)^2 f(u)^2}{\cosh\theta}\, dA,$$

making use of (26), we obtain

$$\int_F \max(-K^*_u, 0)\, dA^*_u \leq \int_F \max(-K^F, 0)\frac{1}{\cosh\theta}\, dA < \int_F \max(-K^F, 0)\, dA < \infty.$$

Therefore, the Riemannian surface (F, g^*_u) is complete and it has finite total curvature.

□

Now, we can state one of main results of Rubio-Salamanca's work.

Theorem 3.2 *Let $M = I \times_f F$ a Lorentzian warped product, with fiber (F, g) a complete Riemannian surface, which has finite total curvature and whose warping function satisfies* $\text{Inf} f > 0$ *and* $\text{Sup} f < \infty$. *If M obeys the NCC, then any entire maximal graph $(\Sigma_u, \langle\, , \,\rangle)$ must be totally geodesic. Moreover, if there exists a point $p \in F$ such that $\frac{K^F(p)}{f(u(p))^2} - (\log f)''(u(p)) > 0$, then u is constant.*

Proof. From previous Lemma, we have that (F, g_u^*) is complete with finite total curvature. Consider the function $\frac{1}{f(u)\cosh\theta}$ on (F, g_u). Then, some computations allow to show that the Laplacian

$$\Delta_u \left(\frac{1}{f(u)\cosh\theta} \right) = -\frac{1}{f(u)^2 \cosh^2\theta} \Delta_u(f(u)\cosh\theta) + 2\frac{|\nabla_u(f(u)\cosh\theta)|^2}{(f(u)^3 \cosh^3\theta)}$$

is non-positive

Taking into account the invariance of superharmonic functions by conformal changes of metric, we get a positive superharmonic function on the complete parabolic Riemannian surface (F, g_u^*) and as a consequence the function $f(u)\cosh\theta$ must be constant. Thus, from the second term of (21), whose expression we will recall,

$$\begin{aligned} \Delta_u(f(u)\cosh\theta) &= \left\{ \frac{K^F}{f(u)^2} - (\log f)''(u) \right\} |\nabla_u u|^2 f(u)\cosh\theta \\ &+ \frac{1}{2}\text{trace}(A^2) f(u)\cosh\theta, \end{aligned}$$

we obtain that the graph $(\Sigma_u, \langle\, , \,\rangle)$ is totally geodesic.

On the other hand, if moreover there exists a point $p \in F$ such that $\frac{K^F(p)}{f(u(p))^2} - (\log f)''(u(p)) > 0$, taking into account the first addend of (21), then there exists an open neighbourhood of $(p, u(p))$ in Σ_u which is contained in the complete maximal graph $u = u_0$, with $f'(u_0) = 0$. As $(\Sigma_u, \langle\, , \,\rangle)$ is entire and totally geodesic, it must to coincide with the totally geodesic spacelike slice $t = u_0$.

□

Of course, this last result extends the classical Calabi-Bernstein theorem in its non-parametric version. Moreover, the extension given by Alías and Albujer [2], [3] for Lorentzian products is also included. On the other hand, the Theorem 3.2 is independent of those given for Lorentzian warped product by Caballero, Romero and Rubio [18], [19].

4 Uniqueness of complete maximal hypersurfaces in spacetimes

In this new section we will make a brief review on some uniqueness results about maximal hypersurfaces in spacetimes. These results can be considered parametric versions of Calabi-Bernstein type problems. On the other hand, these types of problems have both mathematical and physical interest due to their relevance in Mathematical Relativity

The importance in General Relativity of maximal and constant mean curvature spacelike hypersurfaces in spacetimes is well-known; a summary of several reasons justifying it can be found in the paper of Marsden and Tipler [42].

Recall that each maximal hypersurface can describe, in some relevant cases, the transition between the expanding and contracting phases of a relativistic universe. Moreover, they can constitute an initial set for the Cauchy problem [51]. Specifically, Lichnerowicz proved that a Cauchy problem with initial conditions on a maximal hypersurface is reduced to a second-order non-linear elliptic differential equation and a first-order linear differential system [41]. Also, the deep understanding of this kind of hypersurfaces is essential to prove the positivity of the gravitational mass.

On the other hand, they are also interesting for Numerical Relativity, where maximal hypersurfaces are used for integrating forward in time [36]. From a mathematical point of view, it is necessary to study the maximal hypersurfaces of a spacetime in order to understand its structure. Especially, for some asymptotically flat spacetimes, the existence of a foliation by maximal hypersurfaces is established (see for instance, [12] and references therein).

Thus, the existence results and, consequently, uniqueness appear as kernel topics.

Let us remark that the completeness of a spacelike hypersurface is required whenever we study its global properties, and also, from a physical viewpoint, completeness implies that the whole physical space is taken into consideration.

On the other hand, recall that a maximal hypersurface is (locally) a critical point for a natural variational problem, namely of the volume functional (see, for instance [16]). After the relevant result of the Bernstein-Calabi conjecture [21] for the n-dimensional Lorentz-Minkowski spacetime given by Cheng and Yau [22], classical papers dealing with uniqueness results are [17], [27] and [42].

In their work [17], Brill and Flaherty replaced the Lorentz-Minkowski spacetime by a spatially closed universe, and proved uniqueness results for CMC hypersurfaces in the large by assuming $\overline{\mathrm{Ric}}(z,z) > 0$ for every timelike vector z. This assumption may be interpreted as the fact that there is real present matter at every point of the spacetime. It is known as the Ubiquitous Energy Condition.

This energy condition was relaxed by Marsden and Tipler [42] to include, for instance, non-flat vacuum spacetimes.
More recently, Bartnik [12], proved very general existence theorems and consequently, he claimed that it would be useful to find new satisfactory uniqueness results.

Later, Alías, Romero and Sánchez [10], proved new uniqueness results in the class of spacetimes that they called spatially closed Generalized Robertson-Walker (GRW) spacetimes under TCC. Generalized Robertson-Walker spacetimes extend classical Robertson-Walker ones to include the cases in which the fiber has not constant sectional curvature, i.e, they are given as a Lorentzian warped product as we have already described in section 3. Although to be spatially-homogeneous is reasonable as a first approximation of the large scale structure of the universe, this assumption could not be appropriate when we consider a more accurate scale. On the other hand, small deformations of the metric on the fiber of classical Robertson-Walker spacetimes fit into the class of generalized Robertson-Walker spacetimes.

Recall that a spacetime is said spatially closed if it exists a compact spacelike hypersurface in the spacetime. In this work, the authors show that a GRW spacetime is spatially closed if and only its fiber is compact.

Alías, Romero and Sánchez [10], introduce a new technique based on Minkowski-type integral formulas, applying the divergence theorem to the tangent part of the conformal vector field ξ (see, formula 17) on the spacelike hypersurface, as well as, on its image for the shape operator. So, the authors can show that in a spatially closed GRW spacetime obeying the TCC, every compact spacelike hypersurface of constant mean curvature is totally umbilical. In the case of a GRW spacetime $(I \times_f F, \langle \, , \rangle)$, this energy condition is equivalent to following inequalities,

$$f'' \leq 0 \tag{27}$$

and

$$\mathrm{Ric} \geq (n-1)(ff'' - f'^2)\langle \, , \rangle, \tag{28}$$

where Ric denote the Ricci tensor of the n-dimensional fiber (F, g). Taking this equivalence into account, the authors show that in a GRW spacetime satisfying TCC with strict inequality in (28) the only compact spacelike hypersurfaces of constant mean curvature are level spacelike hypersurfaces of the timelike function $t := \pi_I$ (spacelike slices).

In [5], Alías and Montiel using a well-known generalized maximum principle improve the last result aforementioned and prove that in a GRW spacetime whose warping function satisfies the convexity condition $(\log f)'' \leq 0$, the spacelike slices are the only constant mean curvature compact spacelike hypersurfaces.

In 2011, this result was generalized by Caballero, Romero and Rubio [20] for a larger class of spatially closed spacetimes, those who have a gradient conformal timelike vector field. In addition to this, the global structure of this class of spacetimes is analyzed and the relation with its well-known subfamily of generalized RobertsonWalker spacetimes is exposed in detail.

Up to this point, except the Cheng-Yau theorem, all the uniqueness results aforementioned are shown in spatially closed spacetimes. In spite of the historical importance of spatially closed GRW spacetimes, a number of observational and theoretical arguments about the total mass balance of the universe [25] suggest the convenience of taking into consideration open cosmological models. Even more, a spatially closed GRW spacetime violates the holographic principle [15] whereas a GRW spacetime with non-compact fiber could be a suitable model that follows that principle [11]. More precisely, the entropy contained in any spacelike region cannot exceed the area of the regions boundary. That is, if Ω is a compact region of a spacelike hypersurface, and $S(\Omega)$ denotes the entropy of all matter systems in Ω, then

$$S(\Omega) \leq \frac{\mathrm{Area}(\partial\Omega)}{4}.$$

The previous inequality cannot be satisfied by some physical spatially closed spacetimes. For instance, let us consider that a spacetime contains a compact spacelike hypersurface such that it contains a matter system that does not occupy the whole of it. Then, consider a sequence of compact sets contained in the region where no matter system exists, having a point as limit. Using the previous inequality in the outside part of this sequence, we have that the entropy of the whole matter system becomes arbitrarily small, and then it has to be zero. We found a contradiction.

Recently, Romero, Rubio and Salamanca [55] introduce a new class of spatially open GRW spacetimes, which is called spatially parabolic GRW spacetimes. This new notion of spatially parabolic GRW spacetime is a natural counterpart of the spatially closed GRW spacetime. So, a GRW spacetime is spatially parabolic if its fiber is a parabolic Riemannian manifold.

Recall that a complete (non-compact) Riemannian manifold is said to be parabolic if the only positive superharmonic functions are the constants.

On the other hand, the parabolicity of a GRW spacetime's fiber could also be supported by some physical reasons. For instance, galaxies can be understood as molecules (see, for instance, [45, Ch. 12]), if a sonde is sent to the space, its motion may be approached by a Brownian motion, [33]. In fact, the distribution of galaxies and their velocities are not completely known. Parabolicity may favor that the sonde could be observed in any region, since the Brownian motion is recurrent in any parabolic Riemannian manifold [33].

The authors show that under reasonable assumptions on the restriction of the warping function to the spacelike hypersurface and on the boundedness of the hyperbolic angle between the unit normal vector field and the timelike coordinate vector field ∂_t, a complete spacelike hypersurface in a spatially parabolic GRW spacetime is shown to be parabolic, and the existence of a simply connected parabolic spacelike hypersurface in a GRW spacetime also leads to the parabolicity of its fiber. Note that the assumption on the hyperbolic angle of the maximal hypersurface has a physical consequence, this is, relative speed between normal and comoving observers do not approach the speed of light at every point of the hypersurface (see, [60, pp. 45,67]). Then, all the complete maximal hypersurfaces in spatially parabolic GRW spacetimes are determined in several cases, extending, in particular, to this family of open cosmological models several well-known uniqueness results for the case of spatially closed GRW spacetimes (see also [56]).

For arbitrary dimension, parabolicity has no clear relationship with sectional curvature. Indeed, the Euclidean space $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$. Moreover, there exist parabolic Riemannian manifolds whose sectional curvature is not bounded from below.

The family of spatially parabolic GRW spacetimes is very large, although some other interesting GRW spacetimes do not belong to this family. For instance, those Robertson-Walker spacetimes whose fiber is the hyperbolic space $\mathbb{H}^n$ are excluded.

Making use of two maximun principles: the strong Liouville property and the Omori-Yau generalized maximum principle, Romero, Rubio and Salamanca [57] obtain new uniqueness results in other relevant spatially open GRW spacetimes for complete maximal hypersurfaces which are between two spacelike slices (time bounded) and/or have a bounded hyperbolic angle. In contrast to parabolicity, some curvature assumptions should be imposed here.

On the other hand, in the case of the Einstein-de Sitter spacetime, which is a spatially open model, which shows a reasonable fit to recent observations [64], a new uniqueness result for complete maximal (and constant mean curvature spacelike) hypersurfaces is given [58]. The result is obtained applying to the sine of the hyperbic angle of the hypersurface, a Liouville-type theorem (see, [43] and [62]), which is a consequence of the Omori-Yau generalized maximum principle.

Finally, focusing on the problems of uniqueness and non-existence of complete maximal hypersurfaces immersed in a spatially open Robertson-Walker spacetime with flat fiber, Pelegrín, Romero and Rubio [49] give new non-existence and uniqueness results on complete maximal hypersurfaces. Note that these models have aroused a great deal of interest, since recent observations have shown that the current universe is very close to a spatially flat geometry [28].

It is important to say that the authors do not need the hyperbolic angle of the hypersurface to be bounded, which was an assumption used in some previous works studying the spatially open case. Thus, they are able to deal with spacelike hypersurfaces approaching the null boundary at infinity, such as hyperboloids in Minkowski spacetime.

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