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Willmore surfaces and Hopf tori in homogeneous 3-manifolds

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Abstract
Some classification results for closed surfaces in Berger spheres are presented. On the one hand, a Willmore functional for isometrically immersed surfaces into an homogeneous space $\mathbb{E}^3(\kappa, \tau)$ with isometry group of dimension 4 is defined and its first variational formula is computed. Then, we characterize Clifford and Hopf tori as the only Willmore surfaces satisfying a sharp Simons-type integral inequality. On the other hand, we also obtain some integral inequalities for closed surfaces with constant extrinsic curvature in $\mathbb{E}^3(\kappa, \tau)$, becoming equalities if and only if the surface is a Hopf torus in a Berger sphere.

Keywords Willmore surface · Homogeneous space · Constant extrinsic curvature · Clifford torus · Hopf torus

Mathematics Subject Classification 53C42 · 53A10 · 53C30

1 Introduction

A classical problem in the theory of isometric immersions is to classify immersed surfaces into a space form of constant sectional curvature having either constant mean curvature or constant Gaussian curvature. In this direction, we can highlight the rigidity theorems due to Alexandrov [3], Liebmann [18] and Hilbert [14] on surfaces of constant curvature as the most celebrated results in the theory of surfaces in the Euclidean space $\mathbb{R}^3$. For generalizations of these results, we quote [2, 21]. Besides that, we cannot fail to highlight the classical Hopf's theorem [15] which characterizes totally umbilical spheres as the unique topological spheres of constant mean curvature immersed into a three-dimensional space form of constant sectional curvature.

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A natural generalization of space forms is the so-called *homogeneous* spaces. A Riemannian manifold is said to be homogeneous if for any two points p and q, there exists an isometry that maps p into q. Geometrically, an homogeneous manifold seems the same everywhere. As it is well known, simply connected three-dimensional Riemannian homogeneous spaces are classified. Such manifolds have an isometry group of dimension 6, 4 or 3. When the dimension is 6, they correspond to space forms. When the dimension is 3, the manifold has the geometry of the Lie group Sol_3. In the case where the dimension of the isometry group is 4, such manifold fibers over a two-dimensional space form of constant sectional curvature κ, $\mathbb{M}^2(\kappa)$, and its fibers are the trajectories of a unit Killing vector field. These last manifolds are usually denoted by $\mathbb{E}^3(\kappa, \tau)$, where τ is the constant bundle curvature of the natural projection $\pi : \mathbb{E}^3(\kappa, \tau) \to \mathbb{M}^2(\kappa)$ and $\kappa \neq 4\tau^2$. According to the constants κ and τ, we can classify such spaces. When $\tau = 0$, $\mathbb{E}^3(\kappa, 0) = \mathbb{M}^2(\kappa) \times \mathbb{R}$ where $\mathbb{M}^2(\kappa)$ is the sphere $\mathbb{S}^2(\kappa)$ of curvature $\kappa > 0$ or the hyperbolic plane $\mathbb{H}^2(\kappa)$ of curvature $\kappa < 0$. When $\tau \neq 0$, $\mathbb{E}^3(\kappa, \tau)$ is a Berger sphere $\mathbb{S}^3_b(\kappa, \tau)$ if $\kappa > 0$, a Heisenberg group $\mathrm{Nil}_3(\tau)$ if $\kappa = 0$ or the universal cover of $PSL(2, \mathbb{R})$ if $\kappa < 0$.

In the last years, the study of surfaces in homogeneous spaces with 4-dimensional isometry group has attracted the attention of many geometers. We can say that this attention is due to the studies of Abresch, Rosenberg and Meeks which made possible great advances in the research in this area [1, 20, 25]. Indeed, Abresch and Rosenberg [1] discovered an holomorphic quadratic differential for surfaces with constant mean curvature in these spaces and solved the Hopf's theorem for them. Moreover, these spaces are also related to the eight geometries of Thurston [28]. Furthermore, in [12], Gálvez, Martínez and Mira considered the study of the classical Bonnet problem for surfaces in the homogeneous 3-manifolds $\mathbb{E}^3(\kappa, \tau)$. Later on, Rosenberg and Tribuzy showed in [26] a rigidity result for a family of complete surfaces in an homogeneous space having the same positive extrinsic curvature and satisfying a certain condition.

Some years ago, Hu, Lyu and Wang developed in [16] a Simons-type integral inequality for immersed minimal closed surfaces into the homogeneous space $\mathbb{E}^3(\kappa, \tau)$, the equality being satisfied if and only if the surface has parallel second fundamental form. When the homogeneous space $\mathbb{E}^3(\kappa, \tau)$ is the Berger sphere $\mathbb{S}^3_b(\kappa, \tau)$ $(\kappa \neq 4\tau^2)$, they showed that the equality holds if and only if the surface is the Clifford torus. We recall that the Clifford torus is the only minimal Hopf torus in the Berger sphere. Recently, Pámpano has considered in [24] a more general setting, where the ambient space is the total space of a Killing submersion. Specifically, he studies surface energies depending on the mean curvature, which extend the classical notion of Willmore energy. Furthermore, the author constructs critical tori for these energy functionals.

Concerning product spaces, even more recently the second author has studied in [11] immersed complete surfaces into a product space $\mathbb{M}^2(\kappa) \times \mathbb{R}$ with nonnegative constant extrinsic curvature. In this setting, he has shown that these surfaces must be either cylinders when $\kappa = -1$, or slices when $\kappa = 1$. Our goal is, on the one hand, to present a Willmore functional for immersed closed surfaces into $\mathbb{E}^3(\kappa, \tau)$, to obtain its Euler–Lagrange equation, and as a consequence to present a characterization result for closed Willmore surfaces in $\mathbb{S}^3_b(\kappa, \tau)$ in terms of an integral inequality. On the other hand, we extend the techniques developed in [11] to the study of immersed closed surfaces with constant extrinsic curvature into $\mathbb{E}^3(\kappa, \tau)$ $(\tau \neq 0)$.

The outline of the paper goes as follows. In Sect. 2 we describe some basic facts about surfaces in the homogeneous space $\mathbb{E}^3(\kappa, \tau)$ $(\tau \neq 0)$ with isometry group of dimension 4, introducing some relevant families of surfaces in such homogeneous spaces. Later on, working with the Cheng–Yau's operator, we develop in Sect. 3 a Simons-type formula for these surfaces (cf. Proposition 1), as well as a divergence type formula involving the Cheng–Yau's operator (cf. Lemma 2). In Sect. 4 we compute the Euler–Lagrange equation for the Willmore functional of an immersed closed surface into an homogeneous space $\mathbb{E}^3(\kappa,\tau)$ (cf. Proposition 2). As an application, we characterize Clifford and Hopf tori as the only Willmore surfaces satisfying a sharp Simons-type integral inequality (cf. Theorem 1). In the last section, we consider closed surfaces with constant extrinsic curvature and we also obtain integral inequalities, becoming equalities if and only if the surface is a Hopf torus in a Berger sphere $\mathbb{S}^3_b(\kappa,\tau)$ (cf. Theorems 2 and 3).

2 Preliminaries

In this section, we will introduce some basic facts and notations that will appear along the paper.

Let κ and τ be real numbers. The region $\mathcal{D}$ of the Euclidean space $\mathbb{R}^3$ given by

$$\mathcal{D}=\begin{cases}\mathbb{R}^3, & \text{if } \kappa\geq 0\\ \mathbb{D}(2/\sqrt{-\kappa})\times\mathbb{R}, & \text{if } \kappa<0\end{cases}$$

and endowed with the homogeneous Riemannian metric

$$\langle\,,\,\rangle_R=\lambda^2(\mathrm{d}x^2+\mathrm{d}y^2)+(\mathrm{d}z+\tau\lambda(y\mathrm{d}x-x\mathrm{d}y))^2\,,\quad \lambda=\frac{1}{1+\frac{\kappa}{4}(x^2+y^2)},$$

is the so-called *Bianchi–Cartan–Vranceanu space* (BCV-space) which is usually denoted by $\mathbb{E}^3(\kappa,\tau):=(\mathcal{D},\langle\,,\,\rangle_R)$.

As it is well known, there exists a Riemannian submersion $\pi:\mathbb{E}^3(\kappa,\tau)\to\mathbb{M}^2(\kappa)$, where $\mathbb{M}^2(\kappa)$ is the two-dimensional simply connected space form of constant curvature κ, such that π has constant bundle curvature τ and totally geodesic fibers. Furthermore, $\xi=E_3$ is a unit Killing field on $\mathfrak{X}(\mathbb{E}^3(\kappa,\tau))$ which is vertical with respect to π.

The BCV-spaces $\mathbb{E}^3(\kappa,\tau)$ are oriented, and then we can define a vectorial product $\wedge$, such that if $\{e_1,e_2\}$ are linearly independent vectors at a point p, then $\{e_1,e_2,e_1\wedge e_2\}$ determines an orientation at p. Then the properties of ξ imply (see [10]) that for any vector field X on $\mathfrak{X}(\mathbb{E}^3(\kappa,\tau))$ the following relation holds

$$\overline{\nabla}_X\xi=\tau(X\wedge\xi), \tag{1}$$

$\overline{\nabla}$ being the Levi–Civita connection of $\mathbb{E}^3(\kappa,\tau)$. Moreover, let us recall that the curvature tensor of $\mathbb{E}^3(\kappa,\tau)$[1] satisfies, (see [10]),

$$\begin{aligned}\overline{R}(X,Y)Z=&(\kappa-3\tau^2)(\langle X,Z\rangle Y-\langle Y,Z\rangle X)\\&+(\kappa-4\tau^2)\langle Z,\xi\rangle(\langle Y,\xi\rangle X-\langle X,\xi\rangle Y)\\&+(\kappa-4\tau^2)(\langle Y,Z\rangle\langle X,\xi\rangle-\langle X,Z\rangle\langle Y,\xi\rangle)\xi,\end{aligned} \tag{2}$$

where $X,Y,Z\in\mathfrak{X}(\mathbb{E}^3(\kappa,\tau))$.

In what follows, let Σ^2 be an isometrically immersed connected surface which we assume to be orientable and oriented by a globally defined unit normal vector field N. Let us denote by A the second fundamental form of the immersion with respect to N and by ∇ the Levi–Civita connection of Σ^2. Then, the Gauss and Weingarten formulae are given by

[1] We adopt for the $(1,3)$-curvature tensor of the spacetime the following definition ([23,Chapter 3]), $\overline{R}(X,Y)Z=\overline{\nabla}_{[X,Y]}Z-[\overline{\nabla}_X,\overline{\nabla}_Y]Z$.

$$\overline{\nabla}_X Y = \nabla_X Y + \langle A(X), Y\rangle N$$

and

$$A(X) = -\overline{\nabla}_X N,$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$.

Furthermore, we can consider a particular function naturally attached to such a surface Σ^2, namely, $C = \langle N, \xi\rangle$. Let us observe that C measures the cosinus of the angle determined by the vector fields N and ξ. A direct computation shows that the projection of the vector field ξ on $\mathfrak{X}(\Sigma)$ is given by

$$T = \xi^{\top} = \xi - CN, \tag{3}$$

where $(\cdot)^{\top}$ denotes the tangential component of a vector field in $\mathfrak{X}(\mathbb{E}^3(\kappa, \tau))$ along Σ^2. Thus, we get

$$|T|^2 = 1 - C^2. \tag{4}$$

Besides, from (1), (3) and the Gauss and Weingarten formulae we easily obtain the integrability equations,

$$\nabla_X T = C(A - \tau J)(X) \quad \text{and} \quad \nabla C = -(A + \tau J)(T), \tag{5}$$

where J denotes the (oriented) rotation of angle $\pi/2$ on $T\Sigma$ given by $J(X) = N \wedge X$. In particular,

$$\langle J(X), J(Y)\rangle = \langle X, Y\rangle \quad \text{and} \quad J^2(X) = -X,$$

for every $X, Y \in \mathfrak{X}(\Sigma)$. Therefore, from the first equation in (5) it easily follows that

$$\operatorname{div}(T) = 2CH, \tag{6}$$

where div denotes the divergence operator on Σ^2 and H stands for the mean curvature of Σ^2, defined by $H = \frac{1}{2}\operatorname{tr}(A)$. Furthermore, it is immediate to check that

$$4H^2 = |A|^2 + 2K_e, \tag{7}$$

where $|A|^2 = \operatorname{tr}(A^2)$ and $K_e = \det(A)$ denotes the *extrinsic curvature* of Σ^2.

As it is well known, the fundamental equations of Σ^2 are the *Gauss equation*

$$\begin{aligned} R(X,Y)Z = (\kappa - 3\tau^2)(\langle X, Z\rangle Y - \langle Y, Z\rangle X) \\ + (\kappa - 4\tau^2)\langle Z, T\rangle(\langle Y, T\rangle X - \langle X, T\rangle Y) \\ + (\kappa - 4\tau^2)(\langle Y, Z\rangle\langle X, T\rangle - \langle X, Z\rangle\langle Y, T\rangle)T \\ + \langle A(X), Z\rangle A(Y) - \langle A(Y), Z\rangle A(X), \end{aligned} \tag{8}$$

where R denotes the curvature tensor of Σ^2 and $X, Y, Z \in \mathfrak{X}(\Sigma)$, and the *Codazzi equation*

$$\nabla A(X,Y) - \nabla A(Y,X) = (\kappa - 4\tau^2)C(\langle X, T\rangle Y - \langle Y, T\rangle X), \tag{9}$$

where $\nabla A : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ denotes the covariant differential of A,

$$\nabla A(X,Y) = (\nabla_Y A)(X) = \nabla_Y A(X) - A(\nabla_Y X), \quad \text{for all} \quad X, Y \in \mathfrak{X}(\Sigma).$$

From the Gauss equation (8), jointly with (4) and (7), it holds

$$2K = 2\tau^2 + 2(\kappa - 4\tau^2)C^2 + 4H^2 - |A|^2 = 2\tau^2 + 2(\kappa - 4\tau^2)C^2 + 2K_e. \tag{10}$$

Let us recall now some classical surfaces in $\mathbb{E}^3(\kappa, \tau)$ which can be constructed in the following way. Given any regular curve α in $\mathbb{M}^2(\kappa)$, $\pi^{-1}(\alpha)$ is an isometrically immersed surface into $\mathbb{E}^3(\kappa, \tau)$ which is usually known as a *Hopf cylinder*. Hopf cylinders are flat surfaces, which have ξ as a parallel tangent vector field and they are characterized by $C = 0$. Furthermore, these cylinders satisfy

$$H = k_g/2, \quad K = 0, \quad K_e = -\tau^2 \quad \text{and} \quad |\Phi|^2 = 2H^2 + 2\tau^2,$$

where k_g is the geodesic curvature of α.

Moreover, if α is a closed curve and the Riemannian submersion π has circular fibers, which happens just in the case where $\mathbb{E}^3(\kappa, \tau)$ is a Berger sphere $\mathbb{S}^3_b(\kappa, \tau)$, then $\pi^{-1}(\alpha)$ is a flat torus which is also called a *Hopf torus*.

Let us remember at this point that the Berger sphere $\mathbb{S}^3_b(\kappa, \tau)$ is isometric to the usual sphere $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2\,;\ |z|^2 + |w|^2 = 1\}$ endowed with the metric

$$\langle X, Y\rangle = \frac{4}{\kappa}\left(\langle X, Y\rangle_{\mathbb{S}^3} + \frac{1}{\kappa}\left(4\tau^2 - \kappa\right)\langle X, V\rangle_{\mathbb{S}^3}\langle Y, V\rangle_{\mathbb{S}^3}\right),$$

where $\langle\, ,\rangle_{\mathbb{S}^3}$ stands for the usual metric on the sphere, $V_{(z,w)} = J(z, w) = (iz, iw)$ for each $(z, w) \in \mathbb{S}^3$ and κ, τ are real numbers with $\kappa > 0$ and $\tau \neq 0$. We note that if $\kappa = 4\tau^2$ then $\mathbb{S}^3_b(\kappa, \tau)$ is, up to homotheties, the round sphere. The Hopf fibration $\pi : \mathbb{S}^3_b(\kappa, \tau) \to \mathbb{S}^2(\kappa)$, defined by

$$\pi(z, w) = \frac{1}{\sqrt{\kappa}}\left(z\overline{w}, \frac{1}{2}\left(|z|^2 - |w|^2\right)\right),$$

is a Riemannian submersion whose fibers are geodesics. The vertical unit Killing vector field is given by $\xi = \dfrac{\kappa}{4\tau}V$. A particular Hopf torus in $\mathbb{S}^3_b(\kappa, \tau)$ is the *Clifford torus* given by

$$\{(z, w) \in \mathbb{S}^3_b(\kappa, \tau)\,;\ |z|^2 = |w|^2 = 1/2\}.$$

It is well known that the Clifford torus is the only minimal Hopf torus in any Berger sphere (see for instance [30]).

Let us finish this section by recalling a classification result for parallel surfaces in $\mathbb{E}^3(\kappa, \tau)$, proved by Belkhelfa, Dillen and Inoguchi in [6]. From now on, we will understand by a parallel surface a surface with parallel second fundamental form.

Lemma 1 [6,Theorem 8.2] *Let Σ^2 be an isometrically immersed parallel surface into the homogeneous space $\mathbb{E}^3(\kappa, \tau)$, $\kappa - 4\tau^2 \neq 0$. Then,*

1. *if $\tau \neq 0$ Σ^2 is a piece of a Hopf cylinder over a Riemannian circle in $\mathbb{M}^2(\kappa)$, that is, over a closed curve in $\mathbb{M}^2(\kappa)$ with constant geodesic curvature.*
2. *if $\tau = 0$ Σ^2 is either a piece of a slice in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ or of a Hopf cylinder over a Riemannian circle in $\mathbb{M}^2(\kappa)$.*

3 A Simons-type formula for the Cheng–Yau operator in $\mathbb{E}^3(\kappa, \tau)$

In consideration of the foregoing, we are going to compute the Laplacian of $|A|^2$. First and foremost, we recall the following Weitzenböck formula (see for instance [22])

$$\frac{1}{2}\Delta|A|^2 = \frac{1}{2}\Delta\langle A, A\rangle = |\nabla A|^2 + \langle \Delta\, A, A\rangle, \tag{11}$$

where $\Delta A : \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ is the rough Laplacian of the second fundamental form, that is,

$$\Delta A(X) = \operatorname{tr}\left(\nabla^2 A(X, \cdot, \cdot)\right) = \sum_{i=1}^{2} \nabla^2 A(X, e_i, e_i), \tag{12}$$

$\{e_1, e_2\}$ being an orthonormal frame on $\mathfrak{X}(\Sigma)$ and $\nabla^2 A(X, Y, Z) = (\nabla_Z \nabla A)(X, Y)$ for all $X, Y, Z \in \mathfrak{X}(\Sigma)$. In this setting, on the one hand we obtain from the Codazzi equation (9) and the integrability equations (5) the following symmetry in the two firsts variables of $\nabla^2 A$,

$$\begin{aligned} \nabla^2 A(X, Y, Z) = \nabla^2 A(Y, X, Z) &- (\kappa - 4\tau^2)\langle (A + \tau J)(T), Z\rangle \left(\langle X, T\rangle Y - \langle Y, T\rangle X\right) \\ &+ (\kappa - 4\tau^2) C^2 \left(\langle X, (A - \tau J)(Z)\rangle Y - \langle Y, (A - \tau J)(Z)\rangle X\right). \end{aligned} \tag{13}$$

On the other hand, it is not difficult to see that

$$\nabla^2 A(X, Y, Z) = \nabla^2 A(X, Z, Y) + R(Y, Z)A(X) - A(R(Y, Z)X). \tag{14}$$

Making $Y = Z = e_i$ in (13) and taking traces, we have

$$\begin{aligned} \sum_{i=1}^{2} \nabla^2 A(X, e_i, e_i) = \sum_{i=1}^{2} \nabla^2 A(e_i, X, e_i) &- (\kappa - 4\tau^2) C^2 \left(2HX - (A + \tau J)(X)\right) \\ &- (\kappa - 4\tau^2)\left(\langle X, T\rangle (A + \tau J)(T) - \langle A(T), T\rangle X\right). \end{aligned} \tag{15}$$

Furthermore, from (14) it yields

$$\nabla^2 A(e_i, X, e_i) = \nabla^2 A(e_i, e_i, X) + R(X, e_i)A(e_i) - A(R(X, e_i)e_i). \tag{16}$$

Observe now that, using the Gauss equation (8), we get

$$\begin{aligned} \sum_{i=1}^{2} R(X, e_i)A(e_i) = (\kappa - 3\tau^2)(A(X) - 2HX) &- |A|^2 A(X) + A^3(X) \\ &+ (\kappa - 4\tau^2)\left(\langle A(T), T\rangle X - \langle X, T\rangle A(T)\right) \\ &+ (\kappa - 4\tau^2)\left(2H\langle X, T\rangle - \langle A(T), X\rangle\right) T \end{aligned}$$

and

$$\sum_{i=1}^{2} A(R(X, e_i)e_i) = -(\kappa - 3\tau^2)A(X) + A^3(X) - 2HA^2(X) + (\kappa - 4\tau^2)|T|^2 A(X).$$

Thus, inserting these two last equalities in (16),

$$\begin{aligned} \sum_{i=1}^{2} \nabla^2 A(e_i, X, e_i) = \sum_{i=1}^{2} \nabla^2 A(e_i, e_i, X) &+ 2(\kappa - 3\tau^2)(A(X) - HX) + 2HA^2(X) \\ &+ (\kappa - 4\tau^2)\left(\langle A(T), T\rangle X - \langle X, T\rangle A(T) - |T|^2 A(X)\right) \\ &+ (\kappa - 4\tau^2)\left(2H\langle X, T\rangle - \langle A(T), X\rangle\right) T - |A|^2 A(X). \end{aligned} \tag{17}$$

Observe now that, since the trace commutes with the Levi–Civita connection,

$$\sum_{i=1}^{2} \nabla^2 A(e_i, e_i, X) = \operatorname{tr}(\nabla_X \nabla A) = \nabla_X(\operatorname{tr}(\nabla A)).$$

We claim that

$$\mathrm{tr}(\nabla A) = 2\nabla H + C(\kappa - 4\tau^2)T. \tag{18}$$

Indeed, using the Codazzi equation (9),

$$\begin{aligned}\langle \nabla A(e_i, e_i), X\rangle &= \langle (\nabla_{e_i} A)(e_i), X\rangle = \langle e_i, (\nabla_{e_i} A)(X)\rangle = \langle e_i, \nabla A(X, e_i)\rangle \\ &= \langle e_i, \nabla A(e_i, X)\rangle + (\kappa - 4\tau^2)C(\langle X, T\rangle - \langle X, e_i\rangle\langle T, e_i\rangle),\end{aligned}$$

which implies that

$$\begin{aligned}\langle \mathrm{tr}(\nabla A), X\rangle &= \sum_{i=1}^{2} \left[\langle e_i, (\nabla_X A)(e_i)\rangle + C(\kappa - 4\tau^2)\left(\langle X, T\rangle - \langle X, e_i\rangle\langle T, e_i\rangle\right)\right] \\ &= 2\langle \nabla H, X\rangle + (\kappa - 4\tau^2)C\langle X, T\rangle,\end{aligned}$$

for all $X \in \mathfrak{X}(\Sigma)$, so the claim is proved. Hence, from (18) and (5), it holds

$$\nabla_X(\mathrm{tr}(\nabla A)) = 2\nabla_X \nabla H - (\kappa - 4\tau^2)\langle (A + \tau J)(T), X\rangle T + (\kappa - 4\tau^2)C^2(A - \tau J)(X). \tag{19}$$

Therefore, putting (15), (17) and (19) in (12),

$$\begin{aligned}\Delta A(X) &= 2\nabla_X \nabla H + 2(\kappa - 3\tau^2)(A(X) - HX) - |A|^2 A(X) + 2HA^2(X) \\ &\quad + (\kappa - 4\tau^2)\left(2\langle A(T), T\rangle X - 2\langle X, T\rangle A(T) - |T|^2 A(X)\right) \\ &\quad + (\kappa - 4\tau^2)\left(2H\langle X, T\rangle - 2\langle A(T), X\rangle\right) T + 2(\kappa - 4\tau^2)C^2\left(A(X) - HX\right) \\ &\quad - \tau(\kappa - 4\tau^2)\left(\langle J(T), X\rangle T + \langle X, T\rangle J(T)\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\langle \Delta A, A\rangle &= 2\mathrm{tr}(A \circ \mathrm{Hess}\, H) + 2(\kappa - 3\tau^2)(|A|^2 - 2H^2) + 2(\kappa - 4\tau^2)C^2(|A|^2 - 2H^2) \\ &\quad + 2(\kappa - 4\tau^2)\left(3H\langle A(T), T\rangle - 2\langle A^2(T), T\rangle - \tau\langle A(T), J(T)\rangle\right) \\ &\quad - (\kappa - 4\tau^2)|T|^2|A|^2 - |A|^4 + 2H\mathrm{tr}(A^3).\end{aligned} \tag{20}$$

Now, taking into account the characteristic polynomial of A, we observe that

$$4H\langle A(T), T\rangle - 2\langle A^2(T), T\rangle = 2|T|^2 K_e. \tag{21}$$

Besides that, from (7) it holds

$$\begin{aligned}&2(\kappa - 3\tau^2)(|A|^2 - 2H^2) + (\kappa - 4\tau^2)(1 - C^2)(2K_e - |A|^2) \\ &\quad = 2(\kappa - 3\tau^2)(|A|^2 - 2H^2) - 2(\kappa - 4\tau^2)(1 - C^2)(|A|^2 - 2H^2) \\ &\quad = 2(|A|^2 - 2H^2)(\tau^2 + (\kappa - 4\tau^2)C^2).\end{aligned} \tag{22}$$

Moreover, it is easy to check that $\mathrm{tr}(A^3) = 3H|A|^2 - 4H^3$, so again from (7) we deduce that

$$-|A|^4 + 2H\mathrm{tr}(A^3) = -|A|^4 + 6H^2|A|^2 - 8H^4 = 2(|A|^2 - 2H^2)K_e. \tag{23}$$

Hence, taking into account (10), (21), (22) and (23), (20) reads

$$\begin{aligned}\langle \Delta A, A\rangle &= 2\mathrm{tr}(A \circ \mathrm{Hess}\, H) + 2(|A|^2 - 2H^2)K + 2(\kappa - 4\tau^2)C^2(|A|^2 - 2H^2) \\ &\quad + 2(\kappa - 4\tau^2)\left(H\langle A(T), T\rangle - \langle A^2(T), T\rangle - \tau\langle A(T), J(T)\rangle\right),\end{aligned}$$

so (11) yields

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 &= |\nabla A|^2 + 2\mathrm{tr}(A\circ \mathrm{Hess}\, H)\\ &\quad + 2(|A|^2-2H^2)K + 2(\kappa-4\tau^2)C^2(|A|^2-2H^2)\\ &\quad + 2(\kappa-4\tau^2)\left(H\langle A(T),T\rangle - \langle A^2(T),T\rangle - \tau\langle A(T),J(T)\rangle\right).\end{aligned} \tag{24}$$

Remark 1 Let us observe that formula (24) was already obtained in [16]. In fact, let us consider a local orthonormal frame $\{e_1, e_2\}$ such that $A(e_1)=\lambda_1 e_1$ and $A(e_2)=\lambda_2 e_2$. Moreover, by the definition of J, we must have $J(e_1)=e_2$ and $J(e_2)=-e_1$. Taking into account these two facts and writing $T=\langle T,e_1\rangle e_1+\langle T,e_2\rangle e_2$, we have

$$\langle A(T),J(T)\rangle = (\lambda_2-\lambda_1)\langle T,e_1\rangle\langle T,e_2\rangle,$$

so we recover [16,Lemma 3.1]. However, we have included the proof for the sake of completeness, and because it represents an alternative reasoning based on tensorial analysis.

Nevertheless, our aim in this section is to obtain a Simons-type formula for the Cheng–Yau's operator. To this respect, following [9] we introduce the Cheng–Yau's operator $\square$ acting on any smooth function $u:\Sigma^2\to\mathbb{R}$ given by

$$\square u = \mathrm{tr}(P\circ \mathrm{Hess}\, u),$$

where P denotes the first Newton transformation of A, that is, $P:\mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ is the operator given by

$$P = 2HI - A, \tag{25}$$

which is also a self-adjoint linear operator which commutes with A and satisfies $\mathrm{tr}(P)=2H$.

Taking $u=2H$, from equation (7) we obtain the following,

$$\begin{aligned}\square(2H) &= \mathrm{tr}(P\circ \mathrm{Hess}\,(2H))\\ &= 2H\Delta(2H) - 2\mathrm{tr}(A\circ\mathrm{Hess}\, H)\\ &= \frac{1}{2}\Delta(2H)^2 - 4|\nabla H|^2 - 2\mathrm{tr}(A\circ \mathrm{Hess}\, H)\\ &= \frac{1}{2}\Delta|A|^2 + \Delta K_e - 4|\nabla H|^2 - 2\mathrm{tr}(A\circ\mathrm{Hess}\, H).\end{aligned} \tag{26}$$

Inserting (24) in previous equality, we get

$$\begin{aligned}\square(2H) &= \Delta K_e + |\nabla A|^2 - 4|\nabla H|^2 + 2(|A|^2-2H^2)K + 2(\kappa-4\tau^2)C^2\left(|A|^2-2H^2\right)\\ &\quad + 2(\kappa-4\tau^2)\left(H\langle A(T),T\rangle - \langle A^2(T),T\rangle - \tau\langle A(T),J(T)\rangle\right).\end{aligned} \tag{27}$$

For our purpose, it will be more appropriate to deal with the traceless part of A, which is given by $\Phi = A - HI$, with I the identity operator on $\mathfrak{X}(\Sigma)$. Then, $\mathrm{tr}(\Phi)=0$ and

$$|\Phi|^2 = |A|^2 - 2H^2 \geq 0, \tag{28}$$

with equality at $p\in\Sigma^2$ if and only if p is an umbilical point. In contrast to the case where the ambient is a Riemannian product, it was proved in [27] that there does not exist any totally umbilical surface in $\mathbb{E}^3(\kappa,\tau)$ with $\tau\neq 0$.

Now, from the characteristic polynomial of Φ and identity (28), the following equalities hold,

$$-2\langle A^2(T),T\rangle + 2H\langle A(T),T\rangle = -|\Phi|^2|T|^2 - 2H\langle\Phi(T),T\rangle$$

and

$$2C^2|A|^2 - 4H^2C^2 = 2C^2|\Phi|^2.$$

Besides this, equations (4) and (10) give us

$$2K + (\kappa - 4\tau^2)\left(2C^2 - |T|^2\right) = 2K_e + 5(\kappa - 4\tau^2)C^2 - \kappa + 6\tau^2.$$

Therefore, inserting these three last equations in (27), we have finally shown the following Simons-type formula for the Cheng–Yau's operator.

Proposition 1 *Let* Σ^2 *be an isometrically immersed surface into an homogeneous space* $\mathbb{E}^3(\kappa, \tau)$. *Then,*

$$\begin{aligned}\square(2H) = \Delta K_e + |\nabla A|^2 - 4|\nabla H|^2 + |\Phi|^2\left(2K_e + (\kappa - 4\tau^2)(5C^2 - 1) + 2\tau^2\right)\\ - 2(\kappa - 4\tau^2)\left(H\langle\Phi(T), T\rangle + \tau\langle\Phi(T), J(T)\rangle\right).\end{aligned}$$

Remark 2 When $\tau = 0$, as it was said in the Introduction, the homogeneous space $\mathbb{E}^3(\kappa, \tau)$ is exactly the product space $\mathbb{M}^2(\kappa) \times \mathbb{R}$, where $\mathbb{M}^2(\kappa)$ is a space form with constant sectional curvature κ. Thus, Proposition 1 extends [11,Proposition 1.2].

Let us finish this section by showing a nice divergence formula involving the Cheng–Yau's operator.

Lemma 2 *Let* Σ^2 *be an isometrically immersed surface into an homogeneous space* $\mathbb{E}^3(\kappa, \tau)$. *Then,*

$$\mathrm{div}(P(2\nabla H)) = \square(2H) - 2C(\kappa - 4\tau^2)T(H). \tag{29}$$

Proof Observe that by a standard tensor computation

$$\mathrm{div}(P(2\nabla H)) = \square(2H) + 2\langle \mathrm{div}\, P, \nabla H\rangle, \tag{30}$$

where

$$\mathrm{div}\,(P) = \sum_{i=1}^{2} \nabla P(e_i, e_i)$$

with

$$\nabla P(X, Y) = (\nabla_Y P)X = \nabla_Y(PX) - P(\nabla_Y X),$$

for every $X, Y \in \mathfrak{X}(\Sigma)$.

It remains to compute the last term of equation (30). Indeed, from (25),

$$\nabla P(X, Y) = 2Y(H)X - \nabla A(X, Y),$$

for every $X, Y \in \mathfrak{X}(\Sigma)$. Then, (18) implies that

$$\mathrm{div}\,(P) = \mathrm{tr}(\nabla P) = 2\nabla H - 2\nabla H - (\kappa - 4\tau^2)CT = -C(\kappa - 4\tau^2)T, \tag{31}$$

so finally (29) follows from (30) and (31). □

4 Willmore surfaces in $\mathbb{S}^3_b(\kappa, \tau)$

Let $x : \Sigma^2 \to \mathbb{M}^3(\kappa)$ be an isometrically immersed orientable closed, i.e., compact without boundary, surface into the Riemannian space form $\mathbb{M}^3(\kappa)$ with constant sectional curvature κ. The Willmore functional is defined by

$$\mathcal{W}(x) = \int_\Sigma (H^2 + \kappa)\mathrm{d}A,$$

where $\mathrm{d}A$ denotes the area element of the induced metric on Σ^2. Associated to this functional, there is the famous Willmore conjecture, solved in 2012 by Marques and Neves [19], which guarantees that this integral is at least $2\pi^2$ when Σ^2 is an immersed torus into $\mathbb{R}^3$. We say that Σ^2 is a *Willmore surface* if it is a stationary point for the functional $\mathcal{W}$. Moreover, it is well known that $\mathcal{W}$ is a conformal invariant and its Euler–Lagrange equation is given by (see [7, 31])

$$\Delta H + |\Phi|^2 H = 0.$$

For our interests, let $x : \Sigma^2 \to \mathbb{E}^3(\kappa, \tau)$ be an isometrically immersed orientable closed surface into the homogeneous 3-manifold $\mathbb{E}^3(\kappa, \tau)$. Following Weiner [31], we consider the following Willmore functional,

$$\mathcal{W}(x) = \int_\Sigma (H^2 + \overline{K})\mathrm{d}A,$$

where at any $p \in \Sigma^2$, $\overline{K}$ denotes the sectional curvature of $T_p\Sigma$ in $\mathbb{E}^3(\kappa, \tau)$, which following (2)-(4) can be expressed as

$$\overline{K} = \tau^2 + (\kappa - 4\tau^2)C^2. \tag{32}$$

In the following result we obtain the Euler–Lagrange equation of $\mathcal{W}$, extending the result of Weiner [31,Theorem 2.2] for immersed surfaces into the homogeneous space $\mathbb{E}^3(\kappa, \tau)$.

Proposition 2 *Let $x : \Sigma^2 \to \mathbb{E}^3(\kappa, \tau)$ be an isometrically immersed orientable closed surface. Then x is a stationary point of $\mathcal{W}$ if and only if*

$$\Delta H + \big(|\Phi|^2 + (\kappa - 4\tau^2)(1 + C^2)\big)H - 2(\kappa - 4\tau^2)\langle A(T), T\rangle = 0.$$

Proof Let us consider a variation of x, that is, a smooth map $X : (-\varepsilon, \varepsilon) \times \Sigma^2 \to \mathbb{E}^3(\kappa, \tau)$ satisfying that for each $t \in (-\varepsilon, \varepsilon)$, the map $X_t : \Sigma^2 \to \mathbb{E}^3(\kappa, \tau)$, given by $X_t(p) = X(t, p)$, is an immersion and $X_0 = x$. Then, we can compute the first variation of $\mathcal{W}$ along X, that is,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(X_t)\bigg|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (H_t^2 + \overline{K}_t)\mathrm{d}A_t\bigg|_{t=0} \\ &= \int_\Sigma \left(\frac{\mathrm{d}}{\mathrm{d}t}(H_t^2 + \overline{K}_t)\mathrm{d}A_t + (H_t^2 + \overline{K}_t)\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}A_t)\right)\bigg|_{t=0}, \end{aligned} \tag{33}$$

where, for each $t \in (-\varepsilon, \varepsilon)$, H_t and $\overline{K}_t$ stand, respectively, for the mean curvature of Σ^2 and the sectional curvature of $T_p\Sigma$ in $\mathbb{E}^3(\kappa, \tau)$ with respect to the metric induced by X_t, and $\mathrm{d}A_t$ denotes its volume element.

Observe that, on the one hand, the following identity is well known (see for instance [5])

$$2\frac{\mathrm{d}H_t}{\mathrm{d}t}\bigg|_{t=0} = \Delta f + 2\langle\nabla H, Y^\top\rangle + (\overline{\mathrm{Ric}}(N, N) + |A|^2)\, f,$$

$\overline{\mathrm{Ric}}$ being the Ricci curvature tensor of $\mathbb{E}^3(\kappa,\tau)$ and $Y=\frac{\partial X}{\partial t}\big|_{t=0}$ the variational vector field related to the variation X, which can be decomposed as $Y=Y^\top+fN$ with $f=\langle Y,N\rangle$.

On the other hand, denoting by N_t the unit normal vector field along Σ^2 with respect to the metric induced by X_t, since $N_0=N$ it holds

$$\frac{\mathrm{d}\overline{K}_t}{\mathrm{d}t}\bigg|_{t=0}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\tau^2+(\kappa-4\tau^2)\langle N_t,\xi\rangle^2\right)\bigg|_{t=0}=2(\kappa-4\tau^2)\langle N_t,\xi\rangle\frac{\mathrm{d}}{\mathrm{d}t}\langle N_t,\xi\rangle\bigg|_{t=0}.$$

Since $Y=\frac{\partial X}{\partial t}\big|_{t=0}$ is a coordinate field, there exists an orthonormal frame $\{e_1,e_2\}$ in $\mathfrak{X}(\Sigma)$ such that $[Y,e_k]=0$, for any $k=1,2$. Thus, a direct computation gives us

$$\nabla f=-\overline{\nabla}_Y N-A(Y^\top).$$

Then, from the integrability equations (5) we get

$$\frac{\mathrm{d}\overline{K}_t}{\mathrm{d}t}\bigg|_{t=0}=-2(\kappa-4\tau^2)C\left(\langle\nabla f,T\rangle+\langle(A+\tau J)(T),Y^\top\rangle\right).$$

Furthermore, by using Lemma 4.2 of [4] (see also [8,Lemma 5.4]), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}A_t)\bigg|_{t=0}=\left(-2Hf+\mathrm{div}(Y^\top)\right)\mathrm{d}A.$$

Using the previous equalities, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(H_t^2+\overline{K}_t)\bigg|_{t=0}\mathrm{d}A&=2H\frac{\mathrm{d}H_t}{\mathrm{d}t}\bigg|_{t=0}\mathrm{d}A+\frac{\mathrm{d}\overline{K}_t}{\mathrm{d}t}\bigg|_{t=0}\mathrm{d}A\\&=H\left(\Delta f+(\overline{\mathrm{Ric}}(N,N)+|A|^2)f\right)\mathrm{d}A+\langle\nabla H^2,Y^\top\rangle\mathrm{d}A\\&\quad-2(\kappa-4\tau^2)C\left(\langle\nabla f,T\rangle+\langle(A+\tau J)(T),Y^\top\rangle\right)\mathrm{d}A\end{aligned}\tag{34}$$

and

$$(H^2+\overline{K})\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}A_t)\bigg|_{t=0}=-2H(H^2+\overline{K})f\mathrm{d}A+(H^2+\overline{K})\mathrm{div}(Y^\top)\mathrm{d}A.\tag{35}$$

Let us also observe that

$$\mathrm{div}(H^2Y^\top)=H^2\mathrm{div}(Y^\top)+\langle\nabla H^2,Y^\top\rangle.$$

From (5) it also holds

$$\mathrm{div}(\overline{K}Y^\top)=\overline{K}\mathrm{div}(Y^\top)-2(\kappa-4\tau^2)C\langle(A+\tau J)(T),Y^\top\rangle.$$

Then, it follows from (35) that

$$\begin{aligned}(H^2+\overline{K})\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}A_t)\bigg|_{t=0}&=-2H(H^2+\overline{K})f\mathrm{d}A+\mathrm{div}(H^2Y^\top)\mathrm{d}A-\langle\nabla H^2,Y^\top\rangle\mathrm{d}A\\&\quad+\mathrm{div}(\overline{K}Y^\top)\mathrm{d}A+2(\kappa-4\tau^2)C\langle(A+\tau J)(T),Y^\top\rangle\mathrm{d}A.\end{aligned}\tag{36}$$

Hence, replacing (34) and (36) in (33), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(X_t)\Big|_{t=0} &= \int_\Sigma \left(H\Delta f + H(\overline{\mathrm{Ric}}(N,N)+|A|^2)f - 2H(H^2+\overline{K})f\right)\mathrm{d}A\\ &\quad - 2(\kappa-4\tau^2)\int_\Sigma C\langle\nabla f, T\rangle \mathrm{d}A\\ &= \int_\Sigma \left(\Delta H + (\overline{\mathrm{Ric}}(N,N)+|A|^2)H - 2H(H^2+\overline{K})\right) f\mathrm{d}A\\ &\quad - 2(\kappa-4\tau^2)\int_\Sigma C\langle\nabla f, T\rangle \mathrm{d}A.\end{aligned}$$

Besides this, from (5) and (6),

$$\begin{aligned}\mathrm{div}(CfT) &= Cf\,\mathrm{div}(T) + C\langle\nabla f, T\rangle + f\langle\nabla C, T\rangle\\ &= 2HC^2f + C\langle\nabla f, T\rangle - f\langle A(T), T\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(X_t)\Big|_{t=0} &= \int_\Sigma \left(H\Delta f + H(\overline{\mathrm{Ric}}(N,N)+|A|^2)f - 2H(H^2+\overline{K})f\right)\mathrm{d}A\\ &\quad + 2(\kappa-4\tau^2)\int_\Sigma \left(2HC^2 - \langle A(T),T\rangle\right) f\mathrm{d}A\\ &= \int_\Sigma \left(\Delta H + (\overline{\mathrm{Ric}}(N,N)+|A|^2)H - 2H(H^2+\overline{K})\right) f\mathrm{d}A\\ &\quad + 2(\kappa-4\tau^2)\int_\Sigma \left(2HC^2 - \langle A(T),T\rangle\right) f\mathrm{d}A.\end{aligned}$$

Consequently, x is a stationary point of the Willmore functional $\mathcal{W}$ if and only if

$$\Delta H + \left(|\Phi|^2 + \overline{\mathrm{Ric}}(N,N) - 2\overline{K} + 4(\kappa-4\tau^2)C^2\right)H - 2(\kappa-4\tau^2)\langle A(T),T\rangle = 0. \quad (37)$$

Finally, by an straightforward computation from (2) and (4) we easily obtain

$$\overline{\mathrm{Ric}}(N,N) = \kappa - 2\tau^2 - (\kappa-4\tau^2)C^2,$$

which jointly with (32) and (37) yields the desired result. □

Remark 3 It is not difficult to check that minimal surfaces and Hopf cylinders over a curve of geodesic curvature $k_g = \sqrt{2(2\tau^2-\kappa)}$, for all $\kappa, \tau \in \mathbb{R}$ with $\kappa < 2\tau^2$ satisfy (37). So, they are stationary points of the Willmore functional $\mathcal{W}$.

Before presenting our classification result for Willmore surfaces in $\mathbb{E}^3(\kappa,\tau)$, we firstly need the following lemma whose proof follows the ideas developed in [13,Lemma 2.1] (see also [17]).

Lemma 3 *If Σ^2 is an isometrically immersed orientable surface into the homogeneous space $\mathbb{E}^3(\kappa,\tau)$, then*

$$|\nabla A|^2 \geq 3|\nabla H|^2 + 2(\kappa-4\tau^2)C\langle\nabla H, T\rangle. \quad (38)$$

Proof Given any $a \in \mathbb{R}$, let us consider the following tensor

$$\begin{aligned}F(X,Y,Z) &= \langle\nabla A(X,Y), Z\rangle\\ &\quad + a\left(\langle\nabla H, X\rangle\langle Y,Z\rangle + \langle\nabla H, Y\rangle\langle X,Z\rangle + \langle\nabla H, Z\rangle\langle X,Y\rangle\right).\end{aligned}$$

A direct computation gives

$$F(X,Y,Z)^2 = \langle \nabla A(X,Y), Z\rangle^2 + 2aQ_1(X,Y,Z) + a^2 Q_2(X,Y,Z),$$

where

$$Q_1(X,Y,Z) = \langle \nabla A(X,Y), Z\rangle \left(\langle \nabla H, X\rangle\langle Y, Z\rangle + \langle \nabla H, Y\rangle\langle X, Z\rangle + \langle \nabla H, Z\rangle\langle X, Y\rangle\right)$$

and

$$\begin{aligned} Q_2(X,Y,Z) = &\left(\langle \nabla H, X\rangle^2\langle Y, Z\rangle^2 + \langle \nabla H, Y\rangle^2\langle X, Z\rangle^2 + \langle \nabla H, Z\rangle^2\langle X, Y\rangle^2\right) \\ &+ 2\left(\langle \nabla H, X\rangle\langle Y, Z\rangle\langle \nabla H, Y\rangle\langle X, Z\rangle + \langle \nabla H, X\rangle\langle Y, Z\rangle\langle \nabla H, Z\rangle\langle X, Y\rangle\right) \\ &+ 2\langle \nabla H, Y\rangle\langle X, Z\rangle\langle \nabla H, Z\rangle\langle X, Y\rangle. \end{aligned}$$

In order to compute these last terms, let us take $\{e_1, e_2\}$ an orthonormal frame on $\mathfrak{X}(\Sigma)$. Then, it is not difficult to check that

$$\sum_{i,j,k}^{2}\langle \nabla A(e_i,e_j), e_k\rangle^2 = |\nabla A|^2 \quad \text{and} \quad \sum_{i,j,k}^{2} Q_2(e_i,e_j,e_k) = 12|\nabla H|^2.$$

Besides that, from Codazzi equation and (18), we have

$$\begin{aligned} \sum_{i,j,k}^{2} Q_1(e_i,e_j,e_k) &= \sum_{i,j=1}^{2}\left(\langle \nabla A(e_i,e_j), e_j\rangle\langle \nabla H, e_i\rangle + \langle \nabla A(e_i,e_j), e_i\rangle\langle \nabla H, e_j\rangle\right) \\ &\quad + \sum_{i=1}^{2}\langle \nabla A(e_i,e_i), \nabla H\rangle \\ &= 6|\nabla H|^2 + 2(\kappa - 4\tau^2)C\langle \nabla H, T\rangle. \end{aligned}$$

Hence,

$$|F|^2 = |\nabla A|^2 + 2a\left(6|\nabla H|^2 + 2(\kappa - 4\tau^2)C\langle \nabla H, T\rangle\right) + 12a^2|\nabla H|^2.$$

Taking $a = -1/2$ we obtain (38). □

We can finally present our first main result.

Theorem 1 *Let Σ^2 be an isometrically immersed orientable closed Willmore surface into an homogeneous space $\mathbb{E}^3(\kappa,\tau)$. Then,*

$$\begin{aligned} &\int_\Sigma \left(|\Phi|^4 - \left(2\tau^2 - (\kappa - 4\tau^2)(1-3C^2)\right)|\Phi|^2\right) dA \\ &\quad - (\kappa - 4\tau^2)\int_\Sigma \left(|\nabla C|^2 + (K_e + \tau^2)(1-5C^2) + 2\tau^2(1-3C^2)\right) dA \geq 0, \end{aligned}$$

where the equality holds if and only if Σ^2 is a parallel surface.

In particular, if $\kappa < 2\tau^2$ the equality holds if and only if $\mathbb{E}^3(\kappa,\tau) = \mathbb{S}^3_b(\kappa,\tau)$ and Σ^2 is either a Clifford torus or a Hopf torus over a closed curve of geodesic curvature $\sqrt{2(2\tau^2-\kappa)}$ on $\mathbb{S}^2(\kappa)$.

Proof Firstly, taking into account (28), (26) can be written as follows,

$$\begin{aligned} \square(2H) &= 4H\Delta H - 2\mathrm{tr}(A \circ \mathrm{Hess}\, H) \\ &= 2H\Delta H - \frac{1}{2}\Delta|\Phi|^2 - 2|\nabla H|^2 + \frac{1}{2}\Delta|A|^2 - 2\mathrm{tr}(A \circ \mathrm{Hess}\, H), \end{aligned}$$

where $\Delta H^2 = 2H\Delta H + 2|\nabla H|^2$ has been used. Consequently, by (24),

$$\begin{aligned} \Box(2H) = 2H\Delta H &- \frac{1}{2}\Delta|\Phi|^2 + |\nabla A|^2 - 2|\nabla H|^2 \\ &+ |\Phi|^2\left(2K_e + (\kappa - 4\tau^2)(5C^2 - 1) + 2\tau^2\right) \\ &- 2(\kappa - 4\tau^2)\left(H\langle\Phi(T), T\rangle + \tau\langle\Phi(T), J(T)\rangle\right). \end{aligned} \tag{39}$$

Let us observe now that from Lemma 3 we get

$$\begin{aligned} |\nabla A|^2 - 2|\nabla H|^2 &\geq |\nabla H|^2 + 2(\kappa - 4\tau^2)C\langle\nabla H, T\rangle \\ &\geq 2(\kappa - 4\tau^2)C\langle\nabla H, T\rangle, \end{aligned} \tag{40}$$

where the equality holds if and only if

$$|\nabla A|^2 = 3|\nabla H|^2 + 2(\kappa - 4\tau^2)C\langle\nabla H, T\rangle = 0,$$

that is, if and only if Σ^2 is a parallel surface. Then, from Lemma 2 and taking into account (39) and (40) we obtain the following inequality,

$$\begin{aligned} \operatorname{div}(P(2\nabla H)) &= \Box(2H) - 2(\kappa - 4\tau^2)C\langle\nabla H, T\rangle \\ &\geq 2H\Delta H - \frac{1}{2}\Delta|\Phi|^2 + |\Phi|^2\left(2K_e + (\kappa - 4\tau^2)(5C^2 - 1) + 2\tau^2\right) \\ &\quad - 2(\kappa - 4\tau^2)\left(H\langle\Phi(T), T\rangle + \tau\langle\Phi(T), J(T)\rangle\right). \end{aligned}$$

Therefore, the divergence theorem yields

$$\begin{aligned} -2\int_\Sigma H\Delta H \mathrm{d}A \geq &\int_\Sigma |\Phi|^2\left(2K_e + (\kappa - 4\tau^2)(5C^2 - 1) + 2\tau^2\right)\mathrm{d}A \\ &- 2(\kappa - 4\tau^2)\int_\Sigma \left(H\langle\Phi(T), T\rangle + \tau\langle\Phi(T), J(T)\rangle\right)\mathrm{d}A. \end{aligned}$$

On the one hand, from Proposition 2 we can write

$$\begin{aligned} 2H\Delta H &= -2H^2\left(|\Phi|^2 + (\kappa - 4\tau^2)(1 + C^2)\right) + 4(\kappa - 4\tau^2)H\langle A(T), T\rangle \\ &= -\left(|\Phi|^2 + 2K_e\right)\left(|\Phi|^2 + (\kappa - 4\tau^2)(3C^2 - 1)\right) + 4(\kappa - 4\tau^2)H\langle\Phi(T), T\rangle \\ &= -|\Phi|^2\left(2K_e + |\Phi|^2 + (\kappa - 4\tau^2)(3C^2 - 1)\right) - 2(\kappa - 4\tau^2)(3C^2 - 1)K_e \\ &\quad + 4(\kappa - 4\tau^2)H\langle\Phi(T), T\rangle, \end{aligned} \tag{41}$$

where we have used that

$$|\Phi|^2 - 2H^2 = -2K_e, \tag{42}$$

which follows from (7) and (28). Hence,

$$\begin{aligned} 0 \geq &\int_\Sigma |\Phi|^2\left(-|\Phi|^2 + 2(\kappa - 4\tau^2)C^2 + 2\tau^2\right)\mathrm{d}A + 2(\kappa - 4\tau^2)\int_\Sigma (1 - 3C^2)K_e \mathrm{d}A \\ &+ 2(\kappa - 4\tau^2)\int_\Sigma \left(H\langle\Phi(T), T\rangle - \tau\langle\Phi(T), J(T)\rangle\right)\mathrm{d}A. \end{aligned} \tag{43}$$

On the other hand, by using $A^2 - 2HA + K_e I = A^2 - 2H\Phi - \left(|\Phi|^2 + K_e\right)I = 0$ and the integrability equation (5), we easily obtain

$$|\nabla C|^2 = 2H\langle\Phi(T), T\rangle + 2\tau\langle\Phi(T), J(T)\rangle + \left(|\Phi|^2 + K_e + \tau^2\right)|T|^2. \tag{44}$$

Now, let us consider the local orthonormal frame on $\mathfrak{X}(\Sigma)$, $\{e_1, e_2\}$ such that $e_1 = \frac{T}{|T|}$ and $e_2 = J(e_1)$ we get

$$\operatorname{div}(J(T)) = -\langle J(T), \nabla_{e_1} e_1\rangle + e_2(|T|),$$

which using once more the integrability equations (5) yields

$$\operatorname{div}(J(T)) = 2\tau C. \tag{45}$$

So, from (5) and (45),

$$\operatorname{div}(\tau C J(T)) = 2\tau^2 C^2 - \tau\langle (A + \tau J)T, J(T)\rangle = -\tau\langle \phi(T), J(T)\rangle - \tau^2(1 - 3C^2).$$

Thus, by (44)

$$\begin{aligned}2H\langle \Phi(T), T\rangle - 2\tau\langle \phi(T), J(T)\rangle = |\nabla C|^2 + 4\operatorname{div}(\tau C J(T)) + \tau^2(3 - 11C^2)\\ - (|\Phi|^2 + K_e)(1 - C^2)\end{aligned}$$

Therefore, by (43) we obtain

$$\begin{aligned}0 \geq \int_\Sigma |\Phi|^2 \left(-|\Phi|^2 + (\kappa - 4\tau^2)(3C^2 - 1) + 2\tau^2\right) \mathrm{d}A\\ + (\kappa - 4\tau^2)\int_\Sigma \left(|\nabla C|^2 + (K_e + \tau^2)(1 - 5C^2) + 2\tau^2(1 - 3C^2)\right) \mathrm{d}A,\end{aligned} \tag{46}$$

which is the desired inequality.

Moreover, as we have remarked before, the equality holds in (46) if and only if Σ^2 is a closed parallel surface. Then, from Lemma 1 Σ^2 is either a Hopf torus (necessarily in $\mathbb{S}^3_b(\kappa, \tau)$) over a Riemannian circle in $\mathbb{S}^2(\kappa)$, or a piece of a slice in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. However, since Σ^2 is closed this last case only occurs in the case $\tau = 0$ and $\kappa > 0$, which does not satisfy the assumption $\kappa < 2\tau^2$.

Consequently, Σ^2 is a Hopf torus in $\mathbb{S}^3_b(\kappa, \tau)$, so in particular $C = 0$ and $K_e = -\tau^2$. Hence, (41) reads

$$0 = \left(-|\Phi|^2 + 2\tau^2\right)(|\Phi|^2 + \kappa - 4\tau^2).$$

Then, either $|\Phi|^2 = 2\tau^2$, which implies that $H = 0$ and Σ^2 is the Clifford torus, or $|\Phi|^2 + \kappa - 4\tau^2 = 0$. Thus, from (42) we get $H = \sqrt{\frac{2\tau^2-\kappa}{2}}$ and, consequently, Σ^2 is isometric to a Hopf torus in $\mathbb{S}^3_b(\kappa, \tau)$ over a curve of geodesic curvature $\sqrt{2\left(2\tau^2 - \kappa\right)}$ on $\mathbb{S}^2(\kappa)$, for $0 < \kappa < 2\tau^2$. □

5 Classification results for constant extrinsic curvature closed surfaces

Let us begin by obtaining some new interesting divergence formulae, which will play a fundamental role in the proof of the main results in this section.

Lemma 4 *Let Σ^2 be an isometrically immersed surface into the homogeneous space $\mathbb{E}^3(\kappa, \tau)$. Then the following divergence formulae hold on Σ^2,*

(a) $\operatorname{div}(\nabla_T T) = K|T|^2 + T(\operatorname{div} T) + |\nabla T|^2 - 4\tau^2 C^2$.
(b) $\operatorname{div}(\operatorname{div}(T)T) = T(\operatorname{div} T) + 4H^2C^2$.
(c) $\operatorname{div}(|T|\nabla|T|) = K|T|^2 + T(\operatorname{div} T) + |\nabla T|^2 - 2\tau^2|T|^2 - 2\tau\langle \Phi(T), J(T)\rangle$.

Proof Firstly, let us observe that items (a) and (b) have already been proved in [29] (see also [16,Lemma 3.2]). However, we will include the proofs for the sake of completeness.

From the integrability equations (5) it is immediate to check that

$$\mathrm{div}(\nabla_T T) = \mathrm{div}(C(A-\tau J)(T)) = -\langle A^2(T), T\rangle + \tau^2|T|^2 + C\mathrm{div}(A(T)) - \tau C\mathrm{div}(J(T)). \tag{47}$$

On the one hand, given a local orthonormal frame $\{e_1, e_2\}$ on $\mathfrak{X}(\Sigma)$ diagonalizing A, from the Codazzi equation (9) it holds

$$\begin{aligned}\mathrm{div}(A(T)) &= \sum_{i=1}^{2}\langle(\nabla_{e_i}A)(T), e_i\rangle + \sum_{i=1}^{2}\langle A(\nabla_{e_i}T), e_i\rangle \\ &= \mathrm{tr}(\nabla_T A) + C(\kappa-4\tau^2)|T|^2 + \sum_{i=1}^{2}\langle\nabla_{e_i}T, A(e_i)\rangle \\ &= 2T(H) + C(\kappa-4\tau^2)|T|^2 + C|A|^2,\end{aligned} \tag{48}$$

where in the last equality we have used again (5) and the fact that the trace commutes with the Levi–Civita connection.

On the other hand, (6) yields

$$T(\mathrm{div}(T)) = -2H\langle A(T), T\rangle + 2CT(H). \tag{49}$$

Then, taking into account (45), (48) and (49), (47) reads

$$\mathrm{div}(\nabla_T T) = K|T|^2 + T(\mathrm{div}(T)) + C^2|A|^2 - 2\tau^2C^2,$$

where we have used (10) and (21). Finally, item (a) follows by observing that

$$|\nabla T|^2 = \sum_{i,j=1}^{2}\langle\nabla_{e_i}T, e_j\rangle^2 = C^2(|A|^2 + 2\tau^2), \tag{50}$$

for any $\{e_1, e_2\}$ local orthonormal frame on $\mathfrak{X}(\Sigma)$.

Item (b) follows directly from (6).

With respect to item (c), a direct computation from (5) guarantees us that

$$|T|\nabla|T| = C(A+\tau J)(T). \tag{51}$$

Then, taking divergences in (51),

$$\mathrm{div}\,(|T|\,\nabla|T|) = \mathrm{div}(A(T))C + \tau\mathrm{div}(J(T))C + \langle\nabla C, (A+\tau J)(T)\rangle. \tag{52}$$

It is easy to check from (21) and from the integrability equations (5) that

$$\begin{aligned}\langle\nabla C, (A+\tau J)(T)\rangle &= -\langle A^2(T), T\rangle - 2\tau\langle A(T), J(T)\rangle - \tau^2|T|^2 \\ &= -2H\langle A(T), T\rangle + K_e|T|^2 - 2\tau\langle\Phi(T), J(T)\rangle - \tau^2|T|^2.\end{aligned} \tag{53}$$

Then, item (c) follows by inserting (48)–(50) and (53) in (52). □

Bringing all these formulae together, we get the desired divergence-type formulae,

Corollary 1 *Let Σ^2 be an isometrically immersed surface into an homogeneous space $\mathbb{E}^3(\kappa,\tau)$. Then the following divergence formulae hold,*

$$\begin{aligned}\mathrm{div}\,(\mathcal{U}) = \Delta K_e + |\nabla A|^2 - 4|\nabla H|^2 + 2|\Phi|^2\left(K_e + (\kappa-4\tau^2)(4C^2-1) + \tau^2\right) \\ - 2(\kappa-4\tau^2)\left(2H\langle\Phi(T), T\rangle + (K_e-\tau^2)(1-3C^2)\right),\end{aligned} \tag{54}$$

where $\mathcal{U} = P(2\nabla H) + (\kappa - 4\tau^2)\,(\nabla_T T - |T|\nabla|T| + \mathrm{div}\,(T)T)$ *and*

$$\begin{aligned} \mathrm{div}\,(\mathcal{V}) = \Delta K_e + |\nabla A|^2 - 4|\nabla H|^2 + 2|\Phi|^2 \left(K_e + 3(\kappa - 4\tau^2)C^2 + \tau^2\right) \\ - 2(\kappa - 4\tau^2)\left(|\nabla C|^2 - 2(K_e + \tau^2)C^2\right), \end{aligned} \tag{55}$$

where $\mathcal{V} = P(2\nabla H) + (\kappa - 4\tau^2)\,(|T|\nabla|T| + \mathrm{div}(T)T - \nabla_T T)$.

Proof On the one hand, let $\mathcal{U}_1 = \nabla_T T - |T|\nabla|T| + \mathrm{div}\,(T)T$, then from items (a), (b) and (c) of Lemma 4 we can compute

$$\mathrm{div}\,(\mathcal{U}_1) = -2\tau^2(3C^2 - 1) + 2\tau\langle\Phi(T), J(T)\rangle + T(\mathrm{div}(T)) + 4H^2C^2. \tag{56}$$

Then, from (56), (49) and item (d) in Lemma 4 we get

$$\begin{aligned} \mathrm{div}(\mathcal{U}) = \Box(2H) - 2H(\kappa - 4\tau^2)\left(\langle A(T), T\rangle - 2C^2H\right) \\ - 2\tau(\kappa - 4\tau^2)\left(\tau(3C^2 - 1) - \langle\Phi(T), J(T)\rangle\right). \end{aligned}$$

Taking now into account Proposition 1 jointly with (4) and the definition of Φ, we easily deduce

$$\begin{aligned} \mathrm{div}(\mathcal{U}) = \ \Delta K_e + |\nabla A|^2 - 4|\nabla H|^2 + 2|\Phi|^2\left(K_e + (\kappa - 4\tau^2)(4C^2 - 1) + \tau^2\right) \\ + (\kappa - 4\tau^2)\left(-4H\langle\Phi(T), T\rangle + (1 - 3C^2)(2\tau^2 + |\Phi|^2 - 2H^2)\right). \end{aligned}$$

Then (54) follows from (42).

On the other hand, let us observe that

$$\mathcal{V} - \mathcal{U} = 2(\kappa - 4\tau^2)(|T|\nabla|T| - \nabla_T T).$$

Therefore, from (54) and items (a) and (c) of Lemma 4, it holds

$$\begin{aligned} \mathrm{div}(\mathcal{V}) = \Delta K_e + |\nabla A|^2 - 4|\nabla H|^2 + 2|\Phi|^2\left(K_e + (\kappa - 4\tau^2)(4C^2 - 1) + \tau^2\right) \\ + 2(\kappa - 4\tau^2)\left(4\tau^2C^2 - 2\tau^2|T|^2 - 2\tau\langle\Phi(T), J(T)\rangle\right. \\ \left. -2H\langle\Phi(T), T\rangle - (K_e - \tau^2)(1 - 3C^2)\right). \end{aligned}$$

Then, the desired formula (55) follows from (4) and (44), so Corollary 1 is proved. □

In the next results we will approach the case in which the extrinsic curvature is constant and negative. For this, the following lemma is essential.

Lemma 5 *Let* Σ^2 *be an isometrically immersed orientable surface into the homogeneous space* $\mathbb{E}^3(\kappa, \tau)$ *with constant extrinsic curvature* $K_e < 0$. *Then*

$$|\nabla A|^2 \leq 4|\nabla H|^2. \tag{57}$$

In particular, the equality holds if and only if Σ^2 *is a parallel surface.*

Proof Indeed, let $\{e_1, e_2\}$ be a local orthonormal frame which diagonalizes A, that is, $A(e_i) = \lambda_i e_i$, $i = 1, 2$. Then $|\nabla A|^2 = \sum_{i,j=1}^{2}(e_i(\lambda_j))^2$, and by a direct computation we get

$$4|\nabla H|^2 = (e_1(\lambda_1) + e_1(\lambda_2))^2 + (e_2(\lambda_1) + e_2(\lambda_2))^2.$$

Hence,

$$|\nabla A|^2 - 4|\nabla H|^2 = -2(e_1(\lambda_1)e_1(\lambda_2) + e_2(\lambda_1)e_2(\lambda_2)). \tag{58}$$

On the other hand, since $K_e = \lambda_1\lambda_2$ is a negative constant, taking derivatives with respect to e_1 and e_2,

$$0 = e_i(K_e) = e_i(\lambda_1)\lambda_2 + \lambda_1 e_i(\lambda_2), \quad i = 1, 2. \tag{59}$$

Furthermore, $\lambda_1, \lambda_2 \neq 0$, so from (59) it holds

$$e_i(\lambda_1) = -\frac{\lambda_1}{\lambda_2} e_i(\lambda_2), \quad i = 1, 2.$$

Therefore, (58) reads

$$|\nabla A|^2 - 4|\nabla H|^2 = \frac{2\lambda_1}{\lambda_2}(e_1^2(\lambda_2) + e_2^2(\lambda_2)) = \frac{2K_e}{\lambda_2^2}(e_1^2(\lambda_2) + e_2^2(\lambda_2)) \leq 0 \tag{60}$$

as desired. The conclusion about the equality is immediate. □

Corollary 2 *There exists no immersed surface into the homogeneous space $\mathbb{E}^3(\kappa, \tau)$ with $\kappa - 4\tau^2 \neq 0$, satisfying the equality in (57) and having positive constant extrinsic curvature.*

Proof Indeed, suppose there exists an immersed surface Σ^2 into $\mathbb{E}^3(\kappa, \tau)$, $\kappa - 4\tau^2 \neq 0$, satisfying the equality in (57) and having positive constant extrinsic curvature. Following the same reasoning as in the proof of Lemma 5, we obtain (60), so

$$0 = |\nabla A|^2 - 4|\nabla H|^2 = \frac{2K_e}{\lambda_2^2}(e_1^2(\lambda_2) + e_2^2(\lambda_2)) \geq 0.$$

Since $K_e > 0$, we must have $e_1(\lambda_2) = e_2(\lambda_2) = 0$. Therefore, λ_2 is constant, so by the assumption on the extrinsic curvature λ_1 is also constant. Thus, Σ^2 should be a parallel surface of $\mathbb{E}^3(\kappa, \tau)$. Hence, from Lemma 1 Σ^2 is either isometric to a piece of a Hopf cylinder or of a slice, which is a contradiction since in both cases $K_e = -\tau^2 \leq 0$. □

Now, we present our first result related to surfaces with constant extrinsic curvature in $\mathbb{S}_b^3(\kappa, \tau)$.

Theorem 2 *Let Σ^2 be an isometrically immersed closed surface into the homogeneous space $\mathbb{E}^3(\kappa, \tau)$, $\kappa - 4\tau^2 \neq 0$, with negative constant extrinsic curvature. Then*

$$\int_\Sigma |\Phi|^2 \big(K_e + (\kappa - 4\tau^2)(4C^2 - 1) + \tau^2\big)\, \mathrm{d}A \geq (\kappa - 4\tau^2) \int_\Sigma Q_{\tau, K_e} \mathrm{d}A,$$

where

$$Q_{\tau, K_e} = 2H\langle \Phi(T), T\rangle + (K_e - \tau^2)(1 - 3C^2). \tag{61}$$

The equality holds if and only if $\mathbb{E}^3(\kappa, \tau) = \mathbb{S}_b^3(\kappa, \tau)$ and Σ^2 is a Hopf torus over a Riemannian circle in $\mathbb{S}^2(\kappa)$.

Proof By Corollary 1,

$$\begin{aligned} \operatorname{div}(\mathcal{U}) = {} & \Delta K_e + |\nabla A|^2 - 4|\nabla H|^2 + 2|\Phi|^2 \big(K_e + (\kappa - 4\tau^2)(4C^2 - 1) + \tau^2\big) \\ & - 2(\kappa - 4\tau^2)\big(2H\langle \Phi(T), T\rangle + (K_e - \tau^2)(1 - 3C^2)\big). \end{aligned}$$

Since we are supposing that the extrinsic curvature is a negative constant, from Lemma 5, we can estimate the divergence in this way

$$\begin{aligned} \operatorname{div}(\mathcal{U}) \leq {} & 2|\Phi|^2 \big(K_e + (\kappa - 4\tau^2)(4C^2 - 1) + \tau^2\big) \\ & - 2(\kappa - 4\tau^2)\big(2H\langle \Phi(T), T\rangle + (K_e - \tau^2)(1 - 3C^2)\big). \end{aligned}$$

Therefore, taking integrals and using the classical divergence theorem, we have

$$\int_\Sigma \left\{|\Phi|^2\left(K_e+(\kappa-4\tau^2)(4C^2-1)+\tau^2\right)-(\kappa-4\tau^2)Q_{\tau,K_e}\right\}\mathrm{d}A\geq 0, \qquad (62)$$

where Q_{τ,K_e} is defined as in (61), which is the desired inequality.

Furthermore, the equality is satisfied if and only if the equality holds in (57). Since $K_e<0$, Lemma 5 guarantees that Σ^2 is a parallel surface in $\mathbb{E}^3(\kappa,\tau)$. Therefore, from Lemma 1, we conclude that Σ^2 is isometric to a piece of a Hopf cylinder or to a slice of $\mathbb{M}^2(\kappa)\times\mathbb{R}$ when $\tau=0$. Thus, by closedness and recalling that slices in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ are totally geodesic surfaces, so consequently satisfy $K_e=0$, the equality in (62) is only satisfied in the case where $\mathbb{E}^3(\kappa,\tau)=\mathbb{S}^3_b(\kappa,\tau)$ and Σ^2 is isometric to a Hopf torus. □

We can also obtain the following alternative characterization result from (55).

Theorem 3 *Let Σ^2 be an isometrically immersed closed surface with negative constant extrinsic curvature into the homogeneous space $\mathbb{E}^3(\kappa,\tau)$ such that $\kappa-4\tau^2>0$. Then*

$$\int_\Sigma \left\{\left(3(\kappa-4\tau^2)C^2+K_e+\tau^2\right)|\Phi|^2+2(\kappa-4\tau^2)(K_e+\tau^2)C^2\right\}\mathrm{d}A \geq 0. \qquad (63)$$

The equality holds if and only if $\mathbb{E}^3(\kappa,\tau)=\mathbb{S}^3_b(\kappa,\tau)$ and Σ^2 is a Hopf torus over a Riemannian circle in $\mathbb{S}^2(\kappa)$.

Proof The proof of (63) follows immediately taking integrals in (55) and taking into account Lemma 5. The conclusion regarding the equality follows as in Theorem 2. □

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