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INTERACTIONS BETWEEN EPISTEMOLOGIES OF MATHEMATICS AND EDUCATIONAL SYSTEMS - THE EMERGENCE OF MATHEMATICAL COMMUNITIES ACCORDING TO CULTURES AND STATES IN 19TH CENTURY EUROPE[1]

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Abstract

This paper discusses the generally shared conviction of mathematics being a universal science, with a "common language" and a "shared research agenda". These convictions are discussed in particular with regard to assertions in the volume "Mathematics Unbound" of 2002, where it is maintained that national mathematical communities emerged during the 19th century but converged to a universal community during the 20th century. Emphasising the key importance of the national educational structures, it is argued here that national communities emerged already in the wake of Humanism. The differing "languages" for conceiving of negative numbers provide revealing examples for showing epistemologies related to different educational structures. And a fundamentalist "language" in Italy shows the alignment of mathematics education with classicist conceptions of education. Connecting with the conception of "national styles", the paper proposes approaches to understand characteristics marking the differences between national mathematical communities as tied to social and cultural values and revealed by the education systems. In the conclusion, the claim of an emerged international community is discussed.

Keywords: *universal mathematics; national communities; language of mathematics; negative numbers; fundamentalism; communication; transmission; systems theory*

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Interacciones entre epistemologías de las matemáticas y los sistemas educativos: el surgimiento de comunidades matemáticas según las culturas y los estados en la Europa del siglo XIX

Resumen

Este artículo analiza la convicción generalmente compartida de que las matemáticas son una ciencia universal, con un "lenguaje común" y una "agenda de investigación compartida". Estas convicciones se discuten en particular con respecto a las afirmaciones en el volumen "Mathematics Unbound" de 2002, donde se sostiene que las comunidades matemáticas nacionales surgieron durante el siglo XIX pero convergieron en una comunidad universal durante el siglo XX. Al enfatizar la importancia clave de las estructuras educativas nacionales, se argumenta aquí que las comunidades nacionales ya surgieron a raíz del humanismo. Los "lenguajes" diferentes para concebir números negativos proporcionan ejemplos reveladores para mostrar epistemologías relacionadas con diferentes estructuras educativas. Y un "lenguaje" fundamentalista en Italia muestra la alineación de la educación matemática con las concepciones clasicistas de la educación. Conectando con la concepción de "estilos nacionales", el artículo propone enfoques para comprender las características que marcan las diferencias entre las comunidades matemáticas nacionales como las vinculadas a valores sociales y culturales y reveladas por los sistemas educativos. En la conclusión, se discute el reclamo de una comunidad internacional emergente.

Palabras clave*: matemáticas universales; comunidades nacionales; lenguaje de las matemáticas; números negativos; fundamentalismo; comunicación; transmisión; teoría de sistemas*

INTRODUCTION

Mathematics uses to be regarded as a universal discipline, as constituting one whole – being coherent and without fragmentations. This corresponds to conceptualizations in sociology of science; Thomas Kuhn – in his seminal work "The Structure of Scientific Revolutions" of 1962 (Kuhn, 1962) and in later publications –- used to speak of 'scientific community' in the singular, thus assuming the existence of one global community in each scientific discipline and which would thus act as an entirety. If one investigates, however, the emergence of modern mathematics as a discipline from the 19th century, one remarks, for instance, revealing differences between French mathematics, focused on "physico-mathématique", Prussian mathematics, focused on "pure mathematics", and British and Italian mathematics, which even began to prosper with a certain delay.

THE EXISTENCE OF A PLURALITY OF MATHEMATICAL COMMUNITIES

There is, however, more recent research on the history of modern mathematics where one is aware of the parallel existence of a number of mathematical communities. These communities are identified, in fact, as national mathematical communities. I am speaking here in particular of the volume "Mathematics Unbound", edited in 2002 by Karen Parshall and Adrian Rice, where various such national communities are investigated. They clearly affirm: "To date, much historical scholarship has focused on the development of national mathematical community and on national mathematical developments" (Parshall & Rice, 2002, p. 6).

They situate the emergence of such mathematical communities during the period from 1800 onwards and align the process of emergence with the process of professionalisation of mathematics (Parshall & Rice, 2002, p.8). The ensuing process, the internationalisation of mathematics, the constitution of a universal mathematical community proves to be a rather recent process: although initiated already by the International Congresses of Mathematicians, from 1897 on, the definite universalisation is dated from about 1950 on.

Maybe, since the main focus of this volume is to investigate the process of internationalisation of the national communities, there is not much emphasis on analysing what constitutes a national mathematical community, what are its characteristics and by what one such community distinguishes itself from another national mathematical community. The emergence is seen basically as a consequence of a political process, the establishment of nation-states since Modern Times (from 1450 on), their establishment being based "on the political notion of the nation-state" (Parshall & Rice, 2002, p. 10).

The missing analysis of what constitutes a national mathematical community becomes perceivable by the repeated affirmations that there was and is just one mathematics - namely one universal mathematics. The subject matter of mathematics is identified with the "language of mathematics" and this shared language is claimed "to unite mathematicians" world-wide:

There is a distinct supranational, apolitical, intellectual component to this internationalization process, namely, the content of mathematics itself. Mathematicians in national contexts share educational experiences and, hence, research goals and agendas. As mathematics moves beyond national boundaries, these goals and agendas become more universally held. The subject matter – the language of mathematics – comes to unite mathematicians regardless of their national loyalties; the subject matter becomes supranational; it transcends national boundaries altogether. (Parshall & Rice, 2002, p. 10).

In fact, it is just what needs to be studied whether “mathematics moving beyond national boundaries” constitutes such an unproblematic easy process of extension. The extension can only be seen as an easy evolution if the mathematics from other communities follow the same “language”, i.e., are of the same concepts and paradigms, and will not be in conflict with other conceptions and paradigms also moving beyond its boundaries. In fact, Parshall and Rice affirm again the key function of the claimed common language: “Mathematicians, perhaps more than other scientists, developed a common language over the course of the nineteenth century that allowed them to participate in shared research agendas” (Parshall & Rice, 2002, p. 13).

The claim of an unproblematic uniting of mathematics universally seems to parallel an analogous claim of universality of school mathematics. In fact, since the 1980s it was often claimed as evident, that “school mathematics is the same everywhere” (Malaty, 1999). This conviction even seems to be a fundament for the international comparisons like TIMSS and PISA. A closer look shows, however, that maybe the names of the sub-disciplines to be taught might be the same – like algebra and geometry -, but that the conceptions of school mathematics differ considerably, in particular due to different epistemologies.

For better understanding the characteristics of a national mathematical community, and thus to better understand the ultimate process of universalisation, exactly this claim of the language has to be investigated. Let us enter this investigation. But let me first give you an example from the first half of the 19th century that mathematicians in France and in Germany were far from speaking the same language of mathematics.

AN EXAMPLE OF EPISTEMOLOGICAL CONFLICTS BETWEEN THE LANGUAGE OF FRENCH AND GERMAN MATHEMATICS

There was a complete discordance between French mathematics and German mathematics around 1800. Since about 1780, a first mathematical school had been established, the combinatorial school, launched by Carl Hindenburg (1741-1808), mathematics professor at Leipzig University, aimed at establishing a general theory of combinations of any kind – thus trying a complete algebraisation of mathematics and following a programmatic claim of Leibniz Abstracting from particular qualities of elements, it studied all possible forms of their ordered arrangement and establishing new combinations by separation, transposition, permutation, etc., of individual or compound elements.

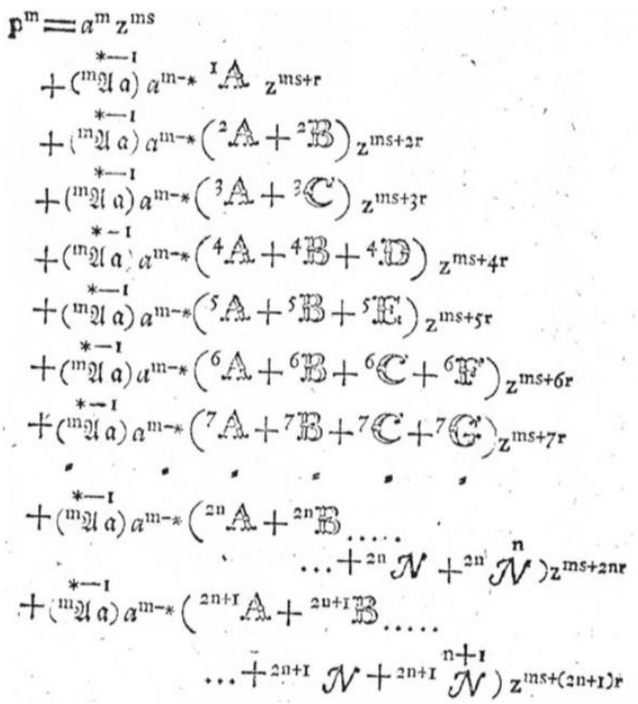

Figure 1. From a text on the polynomial theorem, Hindenburg (1795, p. 397).

The telling title of the key characteristic publication of the school was a paper of 1796: *Der polynomische Lehrsatz, das wichtigste Theorem der ganzen Analysis* – the polynomial theorem, the most important theorem of the entire analysis. This school remained in vigour in Germany until at least the 1830s and dominated the practice of mathematical research (Schubring, 2009, p. 432 f.). In France, it had been entirely rejected. Characteristic is a letter of 1810 from Sylvestre-François Lacroix (1765-1843) to a mathematician in Alsatia who thus was somewhat mediating between France and Germany and who had asked Lacroix for his opinion about the school of combinatorial analysis:

> Analysis and pure geometry are doubtless in themselves very beautiful speculations, quite proper for the exercise of the mind, and they may offer the occasion for the development of much sagacity. But I must confess that I have never been able to attach much importance to these advantages understood as the unique object of the study of these sciences. I have always believed that there were ways of exercising one's reason, and especially of nourishing the activity of one's mind, much more satisfactory than the combination of fatiguing calculations which, when pushed too far, increasingly isolate one from the rest of humanity. After the usual applications, after the 'reasoned' exposition of the major methods, which introduce the philosophy of Science and point out the route for the human spirit to follow in its search for the properties of magnitude, the science of calculation would appear to me to be no more than a sort of game of chess were it not that it offers the key to many phenomena whose laws would be inaccessible without its aid. Therefore I examine each analytic discovery with reference to the hope it may inspire for the advancement of the physico-mathematical sciences. (own translation, quoted from Schubring, 1996, p. 371)

And Lacroix added a somewhat ironical assessment of the same issue by Lagrange: “Il faudrait plutot envelopper que developper" –one needs rather to enwrap than to develop!

The dominance of an understanding of mathematics as “physico-mathématique”, so nicely expressed by Lacroix as dominating in France, turned into a clash for a young German, Edmund Külp (1800-1862), who had studied mathematics in 1819 and 1820 with Alphonse Quetelet (1796-1874), the famous mathematician, astronomer and sociologist, in Brussels according to this epistemological view of physico-mathématique. Back to Germany, he wanted to obtain a doctorate for achieving a university position. He became deeply frightened upon remarking that for this goal he would be forced to work within combinatorial analysis. At Heidelberg he had to follow the mathematics professor Ferdinand Schweins (1780-1856), one of the protagonists of combinatorial analysis:

> Hence I find myself at the University of Heidelberg where I have followed the courses of philosophy and of logic and the lectures on geometry and on analysis […]. Monsieur Schweins, one of the most eminent mathematics professors of this university, absolutely wants me to seriously study combinatorial calculus, which I abhor. (own translation; quoted from Schubring, 2007, p. 111)
>
> I regret more and more to be have been born German. [...] Almost all German mathematicians are exclusively occupied with these calculi, which are so pernicious under all aspects. Please pardon the beginner for daring to speak too boldly. [...] Yes, I am obliged to occupy myself with a terribly voluminous volume of 774 pages, which deals only with this calculus. The result of all this scrupulous research is the discovery of nothing new at all. All what one is proving there is demonstrated [in French mathematics] hundred times more easily and more convenient to the character of the science. (quoted from Schubring, 2007, p. 112).

Eventually, in 1824, Külp had to give up his idea to obtain a doctorate there – his mathematical "language" did not sustain the German combinatorial "language". He became a teacher at secondary schools and is remarkable by having been the mathematics teacher of Georg Cantor in Darmstadt (Schubring, 2007, p.114).

THE EVOLUTION OF A NET OF MATHEMATICAL COMMUNITIES

I am now addressing the emergence of mathematical communities and the analysis of the characteristics of national communities.

In the volume "Mathematics Unbound", Parshall and Rice conceive of Modern Times and in particular what they call "the period of the Scientific Revolution" – for them "roughly from 1450 to 1700" – as the period of building the modern nation-state and consequently as preparing the emergence of national mathematical communities. The preceding period, the Middle Ages, is for them a period of internationalism: they state an essential unity, characterised by the common Catholic theology and the free interchange of studies and communication throughout Europe, based on the common Latin language (Parshall & Rice, 2002, p. 5). They speak therefore of this period as of "transnational universalism". Actually, this implies a too restricted understanding of mathematical communities:

- firstly, although speaking of Europe, in reality they deal with one of its parts, of Western Europe. Until 1453, there had been Eastern Europe, with political centre in Constantinople, but without a comparable development of mathematics. It would deserve another paper to reflect why there was less practice of mathematics in Eastern Rome.

- And secondly, Western Europe was not the only region practicing some mathematics – actually, there were to receive from other regions, with proper conceptions and "languages". There was the well-developed practice in Islamic civilisation, especially in the Maghreb and in Iran; and from the Maghreb initiated the transmission to Western Europe, and from Iran there was communication to India. And while the extensive culture of mathematics in China had for extended periods apparently been autonomous, without communication with other cultures, there occurred communication with India by the second half of the first millennium (of our era).

As a methodological consequence, one has to widen the notion of national mathematical community: as a more general notion, hence, I am proposing "cultural mathematical community".

Socio-political contexts for the emerging national communities

Turning now, with such an already indicated cultural meaning of mathematical communities, to Western Europe, we observe there from the beginning of Modern Times, thus say from 1450 or from 1500, at first a migration of mathematical centres, and then a diversification of mathematical centres. Highly aptly and significant, the term "plurality of algebras" has been coined to express the diversity of algebraic practices in the various areas of 16th century Europe (Rommevaux et al., 2012). By the end of the Middle Ages and by the Renaissance, we have initially Italy as a centre, at first by the commercial developments, but then developing into an algebra; their practice of commercial arithmetic becomes disseminated to other regions in Europe, in particular to Germany. Then there is as an epi-centre in Southern France (the Provence), proving dissemination from Arab Spain. Apparently due to this basis, the centre of mathematical activity eventually migrates from Italy to France. Here, at least by the end of the 17th century, a genuine mathematical community becomes firmly established, thanks in particular to the research structures provided by the *Académie des Sciences* in Paris (see Schubring, 2002, p. 367).

As a matter of fact, we can now observe the emergence of national mathematical communities and this clearly as an effect of the formation of nation-states since the Renaissance, and in particular of the manner how these new manners of governing territories affected education. These manners proved to be different and this markedly along the division into Protestant and Catholic territories since the Protestant Reform and the Catholic Counter-Reform. The differences affected the emergence of the mathematical communities.

The first structural change occurred jointly for all West-European regions: as a part of the Humanism movement during the Renaissance, the sovereigns increasingly took over the control of the universities, so far the only existing definite structures for higher learning but constituting until then closed corporations. As a part of the new state policy, specialised lecturer-ships or professorships for mathematics became instituted within the universities, thus abolishing the former practice of having read mathematical texts by non-specialists, freshly graduated bachelors drawn for this reading by sort. This structural change effected for the first time that the universities became susceptible of providing mathematical specialists who could constitute a community together with others from the same state, i.e. where a homogeneous education system was functioning (Schubring, 2020, pp. 293-294).

Yet, in Catholic territories after the Counter-Reform, where the Jesuits succeeded in taking over universities, they dismantled the traditional arts faculty, dissolved the chairs for mathematics and transformed mathematics, freshly created during Humanism, into a sub-subject of physics in the last grade of their colleges. Thus, in countries and territories remaining Catholic or re-conquered for Catholic faith, conditions for building a mathematical community were rather weak – in particular so in Italy, in Spain, in Portugal, and the Southern states of Germany. In Northern Italy, some states did not admit Jesuits taking over their universities and so there continued professorships for mathematics (Padua, Pisa). France constituted an exceptional case: thanks to the existence of the *Académie des Sciences*, it was a centre for professional mathematicians, producing new knowledge. In general, however, it was for engineering needs (hydraulics for inundation problems, navigational needs, colonial demands) that Catholic states would promote mathematics and thus contribute to establish mathematical communities with strongly applied orientations (Schubring, 2002, p. 367 ff.).

On the other hand, in Protestant countries and territories, the governments even stronger contributed to establish educational systems. In their universities, the professorships for mathematics introduced during Humanism became continued and were even improved. Although the universities continued to serve for teaching, without any obligation for research, some professors began to publish not only textbooks, but also research dissertations. The case of the religiously split Germany is telling: while the southern, Catholic states showed almost no mathematical activity until the 18th century, one remarks consolidated mathematical activity in the northern, Protestant states. Although embracing also all applications in higher education, mathematics there increasingly became oriented towards foundations. In Britain, due to Anglicanism, there was a somewhat mixed situation: the main teaching in universities was given in colleges, by tutors – in general not specialised, but encyclopaedically trained persons – complemented by some lectures besides the regular curriculum, by professors sponsored by some endowment (Schubring, 2002, p. 368 f.p).

As we can see clearly by now: the emergence of national mathematical communities is not due to the 19th century, but already to the 16th and 17th centuries and they are conditioned by the constitution of national educational systems. At the same time, one understands not only differences in the conceptual orientations of these communities; they show themselves also as expressions of different epistemological views of mathematics. In short, the contents are not "supranational"; in fact, I know of no case of "shared research agendas" before the 20th century.

MORE EXAMPLES FOR DIFFERENT "LANGUAGES": THE CONCEPTUAL FIELD OF NEGATIVE NUMBERS

In fact, these structural differences were accompanied by marked differences in methodology and epistemology. While the dominant approach in French textbooks was not to deter beginners ("ne pas rebuter les commençants") and thus to smooth inherent difficulties (Kästner spoke in this context of the "national carelessness of the French" ——(Kästner, 1792, p. 18)), German authors insisted on reflections on the foundations of science. An enormous number of German textbooks published in the second half of the eighteenth century reveals this ambition. Likewise, it is remarkable that Bézout's textbooks, which are so characteristic of the first modernisation of mathematics in the same period within the French military schools, found no contemporary German translators — despite the fact that they were translated into many other European languages. The preface to the German translation of Lazare Carnot's *Reflexions sur la metaphysique du calcul infinitesimal,* published in 1800, contains a revealing example of this attitude. The translator, J. K. F. Hauff, mathematics professor at the University of Marburg, admits that he had first seen an announcement of its French publication in 1797, but that the title had not attracted his attention at the time, "because I […] did not expect much from a metaphysics about geometry by a French author" (Hauff, 1800, p. 1; own translation).

The conceptual field of the negative numbers provides a revealing case of profound differences of epistemologies in various mathematical communities in Europe, from the

18th to the 19th centuries. Negative numbers had become mathematised in different ways in France, in England, and in Germany by the second half of the eighteenth century. By the end of the 17^{th} and by the early 18^{th} centuries, there had been generalizing, algebraizing tendencies in parts of the mathematics communities in England and in France, which had accepted to operate with negative numbers without ontological restrictions – one can name for this tendency Newton in England and the Oratorian mathematicians in France (Schubring, 2005, pp. 88 ff.). But by the middle of the 18^{th} century occurred a rupture in both countries, which made tendencies dominant based on ontological assumptions, which legitimised epistemologically only operations with quantities having some meaning in the real world. Thus, a deep divergence emerged with algebraising developments in Germany. Negative numbers thus became subjects of different mathematical theories and of diverging epistemologies.

In Germany, a theory of *opposite quantities* had emerged and became generally accepted: based on a philosophical notion of *opposition,* quantities were conceived of as provided not only with a quantitative attribute, but also with a second, qualitative one. This qualitative attribute consisted of the possibility of quantities being of the same type and of the same magnitude but of opposed qualities, cancelling out one another. By the turn of the nineteenth century, the legitimising philosophical notion of opposition became mathematised, and opposed quantities were expressed by algebraic notations, like $a+\bar{a}=0$. Wilhelm A. Förstemann, a Prussian *Gymnasium* teacher, summarised these developments and made a step forward by separating the notions of quantity and of number, and by elaborating, in 1817, a coherent theory of negative numbers (Förstemann, 1817).

In France, during the first half of the eighteenth century, there were several approaches to acknowledging negative quantities as legitimate mathematical objects, in particular by real-world interpretations such as debts vs. assets and the like. The process of growing acceptance was stopped by d'Alembert, who campaigned against the use of "isolated" negative quantities, arguing that quantities smaller than "nothing" *(rien)* were contradictory and unacceptable. Assertions like that by Andreas Metz, mathematics professor at Würzburg University, in a textbook typical for the German scene: "It is now easy to understand that $-7 < -3$" (Metz, 1804, p. 53) would have sounded like pure nonsense to d'Alembert. D'Alembert did not differentiate between philosophical notions and mathematical notions like "nothing" and "zero". Since this epistemological stance was widely shared, his conception of negative quantities became influential. Negative solutions of equations were understood as indicators of false assumptions in the hypotheses and as needing correction in order to arrive at positive solutions. This conception of transforming the negative to something positive, determined by a substantialist epistemology of mathematical objects, was in particular applied in textbooks like Bézout's for the military schools (Bézout, 1781). In England, it was in particular two authors who argued from the 1750s vehemently against the existence of negative numbers and against all operations with them: Francis Maseres (1758) and William Frend (1796), thus reducing algebra to arithmetic with natural numbers.

The rejection of negative numbers in France became radicalised, somewhat analogously to England, by Carnot's publications of 1801 and 1803. He reinforced the rejection of negative quantities and tackled this subject as an epistemological question of the relation between algebra and geometry. He denied to algebra all generalising functions, restricting it to a mere translation of geometrically legitimate propositions - and these were essentially interpretable in terms of the real world. Subtraction was accepted only in arithmetic, but not as an algebraic operation. As a consequence, Carnot replaced all notions concerning negative quantities by a geometrical theory, the *geometrie de position,* with a correlation between direct and inverse lines as the basic notion (Schubring, 2005, pp. 353 ff.).

Carnot's reinterpretation of algebra in terms of geometry had a decisive impact on the French view of the architecture of mathematics for a large part of the nineteenth century. Within a few years, his rejection of negative quantities became widely accepted and presented in textbooks.

A revealing indicator of the rupture thus effected is Lacroix's textbook *Elements d'algebre,* in Napoleon's era the only book admitted for this subject in the French secondary schools. In its first two editions (1797 to 1800), Lacroix largely followed Bèzout's model, adopting the ambiguous position of admitting negative quantities as 'real', legitimate objects, but of reinterpreting negative solutions as positive ones. In the third, entirely revised edition (1803), Lacroix replaced all assertions of reality of negative quantities by allusions to the absurdity of negative solutions. Solving equations became now a highly complicated technique and a search for a reinterpretation of the primary assumptions.

A telling example for the different "languages of mathematics" in France and in Germany is provided by Lacroix: since his algebra textbook had to be used also in the German territories annexed to Napoleonic France, Matthias Metternich (1747-1825), mathematics teacher in Mainz, had published a translation. Right in his preface, Metternich emphasised that Lacroix's notions of the signs plus and minus are fluctuating and that his presentation of the different cases of the use of the signs plus and minus lacks mathematical precision. After introducing subtraction, Metternich explains in footnotes that Lacroix's proofs are not rigorous, showing how they have to be transformed in order to arrive at generally valid proofs. Soon, Metternich reaches a point where footnotes no longer suffice; he begins to insert entire paragraphs and even brief chapters in order to introduce a general notion of negative numbers. Consequently, he declared the continued discussion of particular cases in Lacroix's text as "fussily long" and, eventually, ceased translating: "I have ceased translating this long chapter [...] since the reader [after reading my insertions] will no longer doubt the theory of subtraction and of multiplication" (Metternich, 1811, p. 121; my transl.). Thus, the translation in reality was a refutation of the French "language".

The persistence of this epistemologically minded theorising in France is documented, for example, by the 23rd edition of Lacroix's *Eléments d'algèbre* textbook, published by E. Prouhet (Prouhet, 1871), which highly cautiously mentioned in an appendix that negative solutions are admittable - at least to the extent that geometrical interpretations of algebraic concepts were used. The first French textbook, on algebra, exposing negative numbers without reservations was published only in 1896, by Carlo Bourlet.

The conceptual field of negative numbers constitutes an essential element of algebra; differences in its view reveal characteristic differences in epistemology. Another, maybe better-known field of epistemological differences concerns the rigor in analysis and in particular the diverging reception of Weierstraß's famous example of 1872 of a continuous but nowhere differentiable function. The French mathematical community reacted by rejecting such monsters as foreign to sound mathematics. For instance, Gaston Darboux (1842-1917) emphasised in 1875 that practicing such mathematics would endanger the legitimacy of mathematics in France:

> You seem to attach great importance to functions that never have a derivative; for me who am placed in an environment where the kind of studies with which we are concerned are very contested and can only harm those who are working about it, it seems to me that the most considerable step has been taken when one has found continuous functions, which have no derivative for an infinity of values of the independent variable included in the entire interval. What was once admitted and sought to be demonstrated is that every function has a derivative except in exceptional points in a limited number. This idea was overturned by functions such as those by Mr. Schwarz and Hankel. (Schubring, 2012, pp. 571, own translation)

FUNDAMENTALIST "LANGUAGE" IN ITALY

Another revealing case of different "languages", i.e. of different – and here even of diverging – epistemologies is provided by Italy and its adoption of Euclid as official textbook upon establishing the educational system of the eventually united Italian state, in 1867, and the simultaneous rejection of Legendre's geometry: almost against all the other countries and mathematical communities.

Legendre's *Elements de geometrie*, first published in 1794, is the first and important result of the reorientation towards rigour since the French Revolution. Legendre's *Elements* won a distinction from the jury for the *concours* of *livres élémentaires* and the most favourable judgement in mathematics (Schubring, 1989). It was appreciated in this manner: "Monsieur Legendre, in 1794, undertook to revive among us the taste for rigorous demonstration". This geometry textbook turned out to become an international bestseller. From 1802 on, it became translated in at least 13 languages and was re-edited many times, not only in France but also in the other countries, until the end of the 19th century.

- 1794 Original
    - 1802 Italy
        - 1807 Spain
            - 1809 Brazil
                - [1810/1812 Greece]
                - 1819 USA
- 1819 Russia
    - 1822 England
        - 1822 Germany
            - 1826 Sweden
                - 1829 The Netherlands
                - ca. 1830 Switzerland

1836 Ottoman Empire

Figure 2. A list of some of the translations of Legendre's geometry.

Crelle, the translator into German, declared in his preface of 1822: "[It] is distinguished by wealth of content, by clarity, order and consistency of the exposition, by exactness and rigour of the demonstrations" (Crelle, 1822, p. iii).

The first reason for banning in 1867 Legendre and introducing Euclid was nationalism: one wanted to have "genuinely" Italian textbooks – Legendre had been in use in many of the former Italian states (while Euclid was Greek and not Italian …). The second major reason was the intention to achieve an optimal integration of mathematics instruction into the dominant values of Italian secondary schools. These values were then defined by classical languages and literary studies (Scarpis, 1911, p. 27). In the teachers' commentary on the 1867 syllabus, the notion of utility and applicability of mathematical knowledge was denied and replaced by its function to serve as “mental gymnastics to develop the abilities of reason *(raziocinio)*”(Scarpis, 1911, p. 26). Enrico Betti and Francesco Brioschi, the editors of the 1867 edition of Euclid's *Elements* that came into use in the Italian schools, emphasised in their preface the common function of classical languages and of mathematics to serve as "intellectual gymnastics" (Betti & Brioschi, 1867, p. v). In order to comply with this legitimising function, the “harmful confusion” with practical or professional aims in mathematics instruction had to be suppressed, and mathematics had to be “coordinated with the system of classical studies and defined to form an integral part of a common instruction” (Betti & Brioschi, 1867, p. iv). Apparently, classical values were enormously stronger than in any other European country, since “coordination” with these entailed a degree of striving for a ‘purity of method’ which outstripped cultural determinations of school mathematics in the other European countries.

This striving for ‘purity’ leads to the third major reason for the unanimous and flat rejection of Legendre's approach to geometry. In Betti's and Brioschi's preface, the main polemic is directed against Legendre: Euclidean geometry is claimed to constitute a complete science, which is self-sufficient and which does not need support by the science of numbers in any of its demonstrations (Betti & Brioschi, 1867, pp. vi-vii). In fact, the underlying epistemological question was that of the relation between geometry and arithmetic/algebra. Legendre was accused of having mixed both branches in his geometry, making his book unsuitable for the intended methodological instruction.

In all the Italian reflections of this period, the extolling of the educational function ascribed to Greek geometry is coupled with polemics against “mixing” geometry with arithmetic and algebra. While prescribing the Euclidean method as best suited for instilling in pupils the ability to reason rigorously, the instructions for the teachers of 1867 warned against “blurring the purity of ancient geometry by transforming the geometrical theorems into algebraic formula”. It is most characteristic of the underlying mathematical epistemology that geometry was conceived of in exactly the original Greek terms of proportions so that no modernisation by introducing numbers was allowed, and arithmetic remained strictly separated from geometry. The instructions therefore enjoined upon the teachers were to avoid “replacing the concrete magnitudes (lines, angles, surfaces, volumes) by their measures” while emphasising “to reason always on concrete magnitudes, even there where one considers their ratios” (Vita, 1986, p. 7).

MATHEMATICAL COMMUNITIES, COMMUNICATION AND INTERNATIONALISATION

So far, we have seen the rise and functioning of national mathematical communities in Western Europe, with communication still largely restricted to the proper confines until the end of the 19th century. One wonders therefore how these confines could be trespassed and more general communication be established since then so that the present international community of mathematicians could arise.

Which are the basic units for a common understanding of knowledge? My considerations here refer to the sociological theory of science as developed in the theory of systems, in particular by Niklas Luhmann and Rudolf Stichweh, who claim that communication constitutes the basic act of science (Luhmann, 1984; 1990; Stichweh, 1984).

The basic unit sought should thus be constituted by a *common language and a common culture*. These two notions should not be taken too generally, since the same language, for instance, might follow diverging patterns. The features of language and culture should therefore be complemented by that of *nation* or *state*. The interaction between these features occurs essentially within a state's educational system: within the same educational system, it may be reasonably assumed that an educational process extending over many years and the inevitable interactions between representatives of the established culture (and state) and adolescents succeed in constituting commonly accepted methods of attributing meaning and in establishing a shared certain general set of social and cultural values. Within this basic unit thus established, communication may be relatively unproblematic, whereas any step beyond its borders will require new interaction and negotiation for meanings in order to make communication successful.

To refer to 'national styles' seems to mean, in particular, different *epistemological* views. Differences between nations in that respect will usually not concern specific propositions, but rather how these are integrated into the discipline's system of knowledge, what their status is with regard to foundations, how they are interpreted with regard to a philosophy of mathematics, how they are conceived of in the educational curricula, etc. All these issues are contained within the epistemology of the discipline. Since the dominant cultural and social values in a given society and state have been moulded by the specific religious and philosophical traditions influential in its history, it is reasonable to assume a specific relationship between epistemological issues and the national culture in question. It becomes evident how crucial the particular educational system is for establishing typical patterns of communication and for attributing socially shared meanings to concepts – so that, for instance, national mathematical communities can emerge and function. Institutional structures of *schools and of higher education* are materialisations of underlying cultural values and can therefore be used to explore national differences.

We can thus assume that a common understanding will at first be restricted to social communities, which are tied together by certain conditions to form a basic unit of communication, say by sharing a common culture and language. Let us call this basic unit a scientific community of first order. In general, one can assume that these first order communities will share, too, a certain epistemological view of their subject. While there might co-exist different epistemological and conceptual views of mathematics in separate mathematical communities, there should begin processes of interaction at the moment when such separate communities come into contact with each other. Consequently, either the values and conceptions remain mutually alien so that—if there are no other pressures for establishing shared conceptions—the communities will continue to be separated, or a negotiation concerning the differences will begin with the effect of either certain compromises between the two sides or of the domination of one side by the other.

We should thus investigate by which mathematical issues and by which social and political processes communication became at least transnational. Given that one agrees that at least some patterns of internationalisation took place by about 1950, and given also that one agrees that a key factor for this had been the period of Fascism in various European countries, which effected fleeing into exile of an enormous number of mathematicians from these countries and in particular the forced emigration of allegedly Jewish mathematicians from Germany to the United States, it will be productive to remind of the conception of **transmission**. According to applying the notions of *metropolis* and *periphery*, countries where a mathematical community had not yet existed uses to become transmitted a practice of mathematics from some metropolis country. Upon reception, the transmitted system might be adapted and transformed. In the relevant period, from 1800 to 1950, one observes the rise of various mathematical communities in hitherto insofar “underdeveloped” countries by transmission. One revealing case is presented by Brazil, which in 1808 changed from the status of an exploited colony to the mainland of the Portuguese Empire and in 1822 to the Empire of Brazil, developing it science basically by transmission from France. It was in particular the United States, by the last third of the 19th century, by means of introducing graduate colleges – according to the Prussian-German model of Research University and by calling German mathematicians to professorships there. A next such case is presented by Japan, after the Meiji Restoration – also basically moulded by transmission from German mathematics. China seems to be a case of a second-instance transmission: from the United States, after its mathematical emergence due to Germany.

AN OUTLOOK

Since the United States received a second strong transmission from Germany, due to Nazism policy from 1933, one might understand the globalisation of mathematics, the emergence of an international mathematical community after WW II not so much as a consequence of generalised communication via international congresses and journals, but as a product of a multiplied transmission from a few metropolis countries. This does, therefore, not exclude that specific patterns in a number of countries will still persist, even under the conditions of an international community dominated now from a new centre and metropolis.

In fact, given that the emergence of disciplines is intimately tied to the institutionalisation of sciences, the processes of institutionalisation occur within the respective national systems of education and thus according to specific contexts, which can be characterised, on the one hand, by certain epistemologies revealing dominant cultural values and, on the other hand, by structures of that national educational system, which prefigure certain institutional forms and in particular specific embeddings of mathematics within a conception of teaching and research of a certain set of sciences. It is hence quite reasonable to assume that these specific contexts will continue to mould certain particular communities.

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