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New Bounds for Three Outer-Independent Domination-Related Parameters in Cactus Graphs

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Abstract: Let G be a nontrivial connected graph. For a set $D \subseteq V(G)$, we define $\overline{D} = V(G) \setminus D$. The set D is a total outer-independent dominating set of G if $|N(v) \cap D| \geq 1$ for every vertex $v \in V(G)$ and $\overline{D}$ is an independent set of G. Moreover, D is a double outer-independent dominating set of G if $|N[v] \cap D| \geq 2$ for every vertex $v \in V(G)$ and $\overline{D}$ is an independent set of G. In addition, D is a 2-outer-independent dominating set of G if $|N(v) \cap D| \geq 2$ for every vertex $v \in \overline{D}$ and $\overline{D}$ is an independent set of G. The total, double or 2-outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, $\gamma_{\times 2}^{oi}(G)$ or $\gamma_2^{oi}(G)$, is the minimum cardinality among all total, double or 2-outer-independent dominating sets of G, respectively. In this paper, we first show that for any cactus graph G of order $n(G) \geq 4$ with $k(G)$ cycles, $\gamma_2^{oi}(G) \leq \frac{n(G)+l(G)}{2} + k(G)$, $\gamma_t^{oi}(G) \leq \frac{2n(G)-l(G)+s(G)}{3} + k(G)$ and $\gamma_{\times 2}^{oi}(G) \leq \frac{2n(G)+l(G)+s(G)}{3} + k(G)$, where $l(G)$ and $s(G)$ represent the number of leaves and the number of support vertices of G, respectively. These previous bounds extend three known results given for trees. In addition, we characterize the trees T with $\gamma_{\times 2}^{oi}(T) = \gamma_t^{oi}(T)$. Moreover, we show that $\gamma_2^{oi}(T) \geq \frac{n(T)+l(T)-s(T)+1}{2}$ for any tree T with $n(T) \geq 3$. Finally, we give a constructive characterization of the trees T that satisfy the equality above.

Keywords: total outer-independent domination; double outer-independent domination; 2-outer-independent domination; cactus graphs; trees

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1. Introduction

Let $G(V(G), E(G))$ be a finite simple graph of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. For a set $D \subseteq V(G)$, we define $\overline{D} = V(G) \setminus D$ and $N(D) = \cup_{v \in D} N(v)$. As usual, $G - D$ denotes the graph obtained from G by removing all the vertices in D and all the edge incidents with a vertex in D. Analogously, the graph obtained from G by removing all the edges in $\mathcal{U} \subseteq E(G)$ will be denoted by $G - \mathcal{U}$. Given a vertex v of G, $N(v)$ and $N[v]$ represent the *open neighborhood* and the *closed neighborhood* of vertex v; that is, $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg(v)$, is the cardinality of $N(v)$. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. Let $l(G) = |\mathcal{L}(G)|$ and $s(G) = |\mathcal{S}(G)|$, where $\mathcal{L}(G) = \{v \in V(G) : \deg(v) = 1\}$ and $\mathcal{S}(G) = N(\mathcal{L}(G)) \setminus \mathcal{L}(G)$. A *strong leaf* is a leaf at distance two from another leaf. The set of strong leaves is denoted by $\mathcal{L}_s(G)$. A connected graph G is a cactus graph if each edge of G is contained in at most one cycle. We will use the notation P_n, C_n and $K_{1,n-1}$ for the path graphs, cycle graphs and star graphs of order n, respectively. For any other graph theory terminology, we follow the book [1].

Domination in graphs is one of the most popular and highly investigated topics in the area of graph theory. A set $D \subseteq V(G)$ is called a *dominating set* of G if $N(x) \cap D \neq \emptyset$ for every $x \in V(G) \setminus D$. The *domination number* of G, denoted by $\gamma(G)$, is defined as $\gamma(G) = \min\{|D| : D \text{ is a dominating set of } G\}$. A compendium of the main results obtained on domination theory in graphs until 1998 can be found in books [1,2].

In the last decades, one interesting research activity on domination in graphs has been the study of the parameters associated with different variants of dominating sets in graphs. These domination parameters will depend on conditions that can be imposed on the dominating set D, on the set $\overline{D}$ or on the "method" by which vertices in $\overline{D}$ are dominated.

In this article, we study three domination parameters in cactus graphs, which present a certain symmetry due to the conditions that are imposed on each of them. In particular, and as we will show below, these parameters are related to dominating sets whose complements are independent sets, which are nowadays very common research topics in the graph theory community. Next, we define our three domination parameters of interest:

- The total outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality among all *total outer-independent dominating sets* (TOIDSs) of G. In this case, a set $D \subseteq V(G)$ is a TOIDS of G if $|N(v) \cap D| \geq 1$ for every vertex $v \in V(G)$, and $\overline{D}$ is an independent set of G. A $\gamma_t^{oi}(G)$-*set* is a TOIDS of G of cardinality $\gamma_t^{oi}(G)$. This domination parameter was introduced by Soner et al. [3]. For recent results on the total outer-independent domination in graphs, we cite [4–6].
- The double outer-independent domination number of G, denoted by $\gamma_{\times 2}^{oi}(G)$, is the minimum cardinality among all *double outer-independent dominating sets* (DOIDSs) of G. In this case, a set $D \subseteq V(G)$ is a DOIDS of G if $|N[v] \cap D| \geq 2$ for every vertex $v \in V(G)$, and $\overline{D}$ is an independent set of G. A $\gamma_{\times 2}^{oi}(G)$-*set* is a DOIDS of G of cardinality $\gamma_{\times 2}^{oi}(G)$. The study of this domination parameter was initiated in [7]. Some recent and excellent results on this concept can be found, for example, in [6,8,9].
- The 2-outer-independent domination number of G, denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality among all 2-*outer-independent dominating sets* (2OIDSs) of G. In this case, a set $D \subseteq V(G)$ is a 2OIDS of G if $|N(v) \cap D| \geq 2$ for every vertex $v \in \overline{D}$, and $\overline{D}$ is an independent set of G. A $\gamma_2^{oi}(G)$-*set* is a 2OIDS of G of cardinality $\gamma_2^{oi}(G)$. This parameter was introduced in [10] and studied further in [11].

As a consequence of the definitions above, we deduce that if D is a $\gamma_{\times 2}^{oi}(G)$-set or a $\gamma_2^{oi}(G)$-set, then $\mathcal{L}(G) \subseteq D$. In the same way, if W is a $\gamma_t^{oi}(G)$-set or a $\gamma_{\times 2}^{oi}(G)$-set, then $\mathcal{S}(G) \subseteq W$. These facts will be very useful tools throughout the article. Figure 1 shows a cactus graph G with $\gamma_t^{oi}(G) = 3$, $\gamma_{\times 2}^{oi}(G) = 5$ and $\gamma_2^{oi}(G) = 4$.

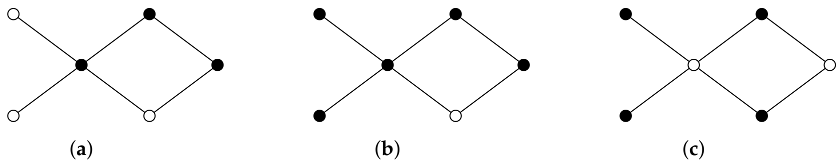

Figure 1. A cactus graph G with $\gamma_t^{oi}(G) = 3$ (**a**), $\gamma_{\times 2}^{oi}(G) = 5$ (**b**) and $\gamma_2^{oi}(G) = 4$ (**c**).

The following theorem, due to Krzywkowski, establishes lower and upper bounds on the previous three outer-independent domination-related parameters for trees.

Theorem 1. *The following bounds hold for any tree T of order $n(T) \geq 4$.*

(i) $\gamma_2^{oi}(T) \leq \frac{n(T)+l(T)}{2}$ *(obtained in* [11]*).*

(ii) $\frac{2n(T)-2l(T)+2}{3} \leq \gamma_t^{oi}(T) \leq \frac{2n(T)-l(T)+s(T)}{3}$ *(obtained in* [12] *and* [13]*, respectively).*

(iii) $\frac{2n(T)+l(T)-s(T)+2}{3} \leq \gamma_{\times 2}^{oi}(T) \leq \frac{2n(T)+l(T)+s(T)}{3}$ *(obtained in* [14] *and* [8]*, respectively).*

In this paper, we first extend the upper bounds given in the previous theorem for the case of the cactus graphs. We prove that for any cactus graph G of order at least four with $k(G)$ cycles,

- $\gamma_2^{oi}(G) \leq \frac{n(G)+l(G)}{2} + k(G)$;
- $\gamma_t^{oi}(G) \leq \frac{2n(G)-l(G)+s(G)}{3} + k(G)$;
- $\gamma_{\times 2}^{oi}(G) \leq \frac{2n(G)+l(G)+s(G)}{3} + k(G)$.

Moreover, we briefly address the particular case of trees. In particular, we characterize the nontrivial trees T with $\gamma_{\times 2}^{oi}(T) = \gamma_t^{oi}(T)$ and we show that $\gamma_2^{oi}(T) \geq \frac{n(T)+l(T)-s(T)+1}{2}$ for any tree T with $n(T) \geq 3$. This previous lower bound covers an existing gap for the 2-outer-independent domination number of a tree, in relation to the other two parameters of interest (see the lower bounds given in Theorem 1 for $\gamma_t^{oi}(T)$ and $\gamma_{\times 2}^{oi}(T)$). Finally, we give a constructive characterization of the trees that satisfy the equality above.

2. New Upper Bounds

We begin this section by extending the upper bound given in Theorem 1-(i) for the case of the cactus graphs. Before, let us recall that $\gamma_2^{oi}(C_n) = \lceil \frac{n}{2} \rceil$ for any cycle C_n of order $n \geq 3$.

Theorem 2. *If G is a nontrivial cactus graph with $k(G)$ cycles, then*

$$\gamma_2^{oi}(G) \leq \frac{n(G)+l(G)}{2} + k(G).$$

Proof. Let G be a nontrivial cactus graph. The proof is by induction on the size $m(G) \geq 1$. If $m(G) \in \{1,2\}$, then it is straightforward that the result follows. These establish the base cases. Let us assume that $m(G) \geq 3$ and that $\gamma_2^{oi}(G^*) \leq (n(G^*) + l(G^*))/2 + k(G^*)$ for each nontrivial cactus graph G^* with $m(G^*) < m(G)$. If G is a tree of order at least four, then the result follows by Theorem 1-(i). On the other hand, if G is a cycle of order $n \geq 3$, then the result follows by the fact that $\gamma_2^{oi}(C_n) = \lceil \frac{n}{2} \rceil$. Henceforth, we will assume that G is a cactus graph other than a cycle or a tree. Hence, G contains at least one cycle as a proper subgraph. Let C be any cycle of G. Also, let $u \in V(C)$ such that $\deg(u) \geq 3$ and $v \in N(u) \cap V(C)$. Let $G' = G - \{uv\}$. Observe that G' is a cactus graph with $n(G') = n(G)$, $k(G') = k(G) - 1$ and $m(G') < m(G)$. By the induction hypothesis, we have the inequality $\gamma_2^{oi}(G') \leq (n(G') + l(G'))/2 + k(G')$. In addition, we observe that $l(G') = l(G)$ or $l(G') = l(G) + 1$. Now, we can distinguish two cases as follows:

Case 1: $l(G') = l(G) + 1$. It is easy to see that $v \in \mathcal{L}(G')$. Let D' be a $\gamma_2^{oi}(G')$-set. Since $\mathcal{L}(G') \subseteq D'$, it follows that $v \in D'$. As a consequence, we deduce that D' is also a 2OIDS of G. Hence, $\gamma_2^{oi}(G) \leq |D'| = \gamma_2^{oi}(G')$. Therefore, by the inequality above and the induction hypothesis, we obtain the following desired result:

$$\begin{aligned}
\gamma_2^{oi}(G) &\leq \gamma_2^{oi}(G') \\
&\leq \frac{n(G')+l(G')}{2} + k(G') \\
&\leq \frac{n(G)+(l(G)+1)}{2} + k(G) - 1 \\
&< \frac{n(G)+l(G)}{2} + k(G).
\end{aligned}$$

Case 2: $l(G') = l(G)$. Let D' be a $\gamma_2^{oi}(G')$-set. Observe that $D = D' \cup \{u\}$ is a 2OIDS of G, which implies that $\gamma_2^{oi}(G) \leq |D| \leq |D'| + 1 \leq \gamma_2^{oi}(G') + 1$. Thus, by the inequality above and the induction hypothesis, we obtain the following desired result:

$$\begin{aligned}\gamma_2^{oi}(G) &\leq \gamma_2^{oi}(G') + 1 \\ &\leq \frac{n(G') + l(G')}{2} + k(G') + 1 \\ &= \frac{n(G) + l(G)}{2} + k(G).\end{aligned}$$

From the two cases above, the proof follows. □

The bound given in the theorem above is tight. For instance, it is achieved for the nontrivial trees attaining the upper bound given in Theorem 1-(i) (see [11]).

In the following result, we extend the upper bound given in Theorem 1-(ii) for the case of the cactus graphs. Before, let us recall that $\gamma_{\times 2}^{oi}(C_n) = \gamma_t^{oi}(C_n) = \lceil \frac{2n}{3} \rceil$ for any cycle C_n of order $n \geq 3$.

Theorem 3. *If G is a cactus graph of order at least four with $k(G)$ cycles, then*

$$\gamma_t^{oi}(G) \leq \frac{2n(G) - l(G) + s(G)}{3} + k(G).$$

Proof. Let G be a cactus graph of order at least four. We proceed by induction on the size $m(G) \geq 3$. If $m(G) = 3$, then G is either the path P_4 or the star $K_{1,3}$. In both cases, the inequality holds by using the upper bound given in Theorem 1-(ii). This establishes the base case. Let us assume that $m(G) \geq 4$ and that $\gamma_t^{oi}(G^*) \leq (2n(G^*) - l(G^*) + s(G^*))/3 + k(G^*)$ for each cactus graph G^* of order at least four such that $3 \leq m(G^*) < m(G)$.

If G is a tree or a cycle, then the result follows by Theorem 1-(ii) or by the fact that $\gamma_t^{oi}(C_n) = \lceil \frac{2n}{3} \rceil$, respectively. Henceforth, we will assume that G is a cactus graph other than a cycle or a tree. Hence, G contains at least one cycle as a proper subgraph. Let C be any cycle of G. Also, let $u \in V(C)$ such that $\deg(u) \geq 3$ and $v \in N(u) \cap V(C)$. Let $G' = G - \{uv\}$. Observe that G' is a cactus graph with $n(G') = n(G)$, $k(G') = k(G) - 1$ and $m(G') < m(G)$. By the induction hypothesis, we have the inequality $\gamma_t^{oi}(G') \leq (2n(G') - l(G') + s(G'))/3 + k(G')$. Now, we proceed to show that $\gamma_t^{oi}(G) \leq \gamma_t^{oi}(G') + 1$. For this purpose, we consider a $\gamma_t^{oi}(G')$-set D'. Observe that $D = D' \cup \{u\}$ is a TOIDS of G, which implies that $\gamma_t^{oi}(G) \leq |D| \leq |D'| + 1 = \gamma_t^{oi}(G') + 1$, as desired. In addition, we observe that $l(G') = l(G)$ or $l(G') = l(G) + 1$. Now, we can distinguish two cases as follows:

Case 1: $l(G') = l(G) + 1$. It is easy to check that $s(G') \leq s(G) + 1$. Therefore, by the previous inequalities and the induction hypothesis, we obtain the following desired result:

$$\begin{aligned}\gamma_t^{oi}(G) \leq \gamma_t^{oi}(G') + 1 &\leq \frac{2n(G') - l(G') + s(G')}{3} + k(G') + 1 \\ &\leq \frac{2n(G) - (l(G) + 1) + (s(G) + 1)}{3} + k(G) \\ &= \frac{2n(G) - l(G) + s(G)}{3} + k(G).\end{aligned}$$

Case 2: $l(G') = l(G)$. In this case, it follows that $s(G') = s(G)$. Thus, by the inequality above and the induction hypothesis, we obtain the following desired result:

$$\begin{aligned}\gamma_t^{oi}(G) \le \gamma_t^{oi}(G') + 1 &\le \frac{2n(G') - l(G') + s(G')}{3} + k(G') + 1 \\ &= \frac{2n(G) - l(G) + s(G)}{3} + k(G).\end{aligned}$$

From the two cases above, the proof follows. □

The bound given in the theorem above is tight. For instance, it is achieved for the nontrivial trees attaining the upper bound given in Theorem 1-(ii) (see [13]).

Next, we extend the upper bound given in Theorem 1-(iii) for the case of the cactus graphs.

Theorem 4. *If G is a cactus graph of order at least three with $k(G)$ cycles, then*

$$\gamma_{\times 2}^{oi}(G) \le \frac{2n(G) + l(G) + s(G)}{3} + k(G).$$

Proof. Let G be a cactus graph with $n(G) \ge 3$. We proceed by induction on the size $m(G) \ge 2$. If $m(G) \in \{2, 3\}$, then it is easy to check that the result follows. These establish the base cases. We assume that $m(G) \ge 4$ and that $\gamma_{\times 2}^{oi}(G^*) \le (2n(G^*) + l(G^*) + s(G^*))/3 + k(G^*)$ for each cactus graph G^* of order $n(G^*) \ge 3$ such that $m(G^*) < m(G)$.

If G is a tree or a cycle, then the result follows by Theorem 1-(iii) or by the fact that $\gamma_{\times 2}^{oi}(C_n) = \lceil \frac{2n}{3} \rceil$, respectively. Henceforth, we will assume that G is a cactus graph other than a cycle or a tree. Hence, G contains at least one cycle as a proper subgraph. Let C be any cycle of G. Also, let $u \in V(C)$ such that $\deg(u) \ge 3$ and $v \in N(u) \cap V(C)$. Let $G' = G - \{uv\}$. Observe that G' is a cactus graph with $n(G') = n(G)$, $k(G') = k(G) - 1$ and $m(G') < m(G)$. By the induction hypothesis, we have the inequality $\gamma_{\times 2}^{oi}(G') \le (2n(G') + l(G') + s(G'))/3 + k(G')$. In addition, we observe that $l(G') = l(G)$ or $l(G') = l(G) + 1$. Now, we can distinguish two cases as follows:

Case 1: $l(G') = l(G) + 1$. It is easy to check that $s(G') \le s(G) + 1$ and that $v \in \mathcal{L}(G')$. Let D' be a $\gamma_{\times 2}^{oi}(G')$-set. Since $\mathcal{L}(G') \subseteq D'$, it follows that $v \in D'$. As a consequence, we deduce that D' is also a DOIDS of G. Hence, $\gamma_{\times 2}^{oi}(G) \le |D'| = \gamma_{\times 2}^{oi}(G')$. Therefore, by the induction hypothesis and the previous inequalities, we obtain the following desired result:

$$\begin{aligned}\gamma_{\times 2}^{oi}(G) &\le \gamma_{\times 2}^{oi}(G') \\ &\le \frac{2n(G') + l(G') + s(G')}{3} + k(G') \\ &\le \frac{2n(G) + (l(G) + 1) + (s(G) + 1)}{3} + k(G) - 1 \\ &< \frac{2n(G) + l(G) + s(G)}{3} + k(G).\end{aligned}$$

Case 2: $l(G') = l(G)$. Observe that $s(G') = s(G)$. Now, let D' be a $\gamma_{\times 2}^{oi}(G')$-set. Observe that $D = D' \cup \{u\}$ is a DOIDS of G, which implies that $\gamma_{\times 2}^{oi}(G) \le |D| \le |D'| + 1 = \gamma_{\times 2}^{oi}(G') + 1$. Hence, by the inequalities above and the induction hypothesis, we obtain the following desired result:

$$\begin{aligned}\gamma_{\times 2}^{oi}(G) &\le \gamma_{\times 2}^{oi}(G') + 1 \\ &\le \frac{2n(G') + l(G') + s(G')}{3} + k(G') + 1\end{aligned}$$

$$= \frac{2n(G) + l(G) + s(G)}{3} + k(G).$$

From the two cases above, the proof follows. □

The bound given in the theorem above is tight. For instance, it is achieved for the nontrivial trees attaining the upper bound given in Theorem 1-(iii) (see [8]).

The Particular Case of Trees

In this subsection, we first address that gap for the 2-outer-independent domination number of a tree of order at least three.

Theorem 5. *For any tree T of order $n(T) \geq 3$,*

$$\gamma_2^{oi}(T) \geq \frac{n(T) + l(T) - s(T) + 1}{2}.$$

Proof. Let T be a tree of order at least three. We proceed by induction on the order $n(T) \geq 3$. If $n(T) \in \{3, 4\}$, then the result follows. These establish the base cases. We assume that $n(T) \geq 5$ and that $\gamma_2^{oi}(T^*) \geq (n(T^*) + l(T^*) - s(T^*) + 1)/2$ for each tree T^* with $3 \leq n(T^*) < n(T)$. Let $v_1 \cdots v_d v_{d+1}$ be a diametral path in T, and we consider the following three cases:

Case 1: $\deg(v_d) \geq 3$. Let us consider the subtree $T' = T - \{v_{d+1}\}$. It is straightforward that $n(T') = n(T) - 1$, $l(T') = l(T) - 1$ and $s(T') = s(T)$. Let D be a $\gamma_2^{oi}(T)$-set. As $\mathcal{L}(T) \subseteq D$ and $\overline{D}$ is an independent set of T, we have that $v_{d+1}, v_{d-1} \in D$, $v_d \in \overline{D}$ and $|(N(v_d) \cap D) \setminus \{v_{d+1}\}| \geq 2$. By the previous conditions, it is easy to deduce that $D \setminus \{v_{d+1}\}$ is a 2OIDS of T', which implies that $\gamma_2^{oi}(T') \leq |D \setminus \{v_{d+1}\}| = \gamma_2^{oi}(T) - 1$. Therefore, by the induction hypothesis and the previous inequalities, we deduce the required result:

$$\begin{aligned}
\gamma_2^{oi}(T) &\geq \gamma_2^{oi}(T') + 1 \\
&\geq \frac{n(T') + l(T') - s(T') + 1}{2} + 1 \\
&= \frac{(n(T) - 1) + (l(T) - 1) - s(T) + 1}{2} + 1 \\
&= \frac{n(T) + l(T) - s(T) + 1}{2}.
\end{aligned}$$

Case 2: $\deg(v_d) = 2$ and $\deg(v_{d-1}) \geq 3$. Let us consider the subtree $T'' = T - \{v_d, v_{d+1}\}$. Let D be a $\gamma_2^{oi}(T)$-set such that $|D \cap \{v_d, v_{d+1}\}|$ is minimum. As $\mathcal{L}(T) \subseteq D$ and $\overline{D}$ is an independent set of T, we have that $v_{d+1}, v_{d-1} \in D$ and $v_d \in \overline{D}$. By the previous conditions, it is easy to deduce that $D \setminus \{v_{d+1}\}$ is a 2OIDS of T'', which implies that $\gamma_2^{oi}(T'') \leq |D \setminus \{v_{d+1}\}| = \gamma_2^{oi}(T) - 1$. Therefore, by the previous inequality, the induction hypothesis and the fact that $n(T'') = n(T) - 2$, $l(T'') = l(T) - 1$ and $s(T'') = s(T) - 1$, we deduce the desired result:

$$\begin{aligned}
\gamma_2^{oi}(T) &\geq \gamma_2^{oi}(T'') + 1 \\
&\geq \frac{n(T'') + l(T'') - s(T'') + 1}{2} + 1 \\
&= \frac{(n(T) - 2) + (l(T) - 1) - (s(T) - 1) + 1}{2} + 1 \\
&= \frac{n(T) + l(T) - s(T) + 1}{2}.
\end{aligned}$$

Case 3: $\deg(v_d) = \deg(v_{d-1}) = 2$. Let us consider the subtree $T'' = T - \{v_d, v_{d+1}\}$. As in the previous case, it can be deduced that $\gamma_2^{oi}(T'') \leq \gamma_2^{oi}(T) - 1$. Therefore, by the previous inequality, the induction hypothesis and the fact that $n(T'') = n(T) - 2$, $l(T'') = l(T)$ and $s(T'') \leq s(T)$, we deduce the required result:

$$\begin{aligned}
\gamma_2^{oi}(T) &\geq \gamma_2^{oi}(T'') + 1 \\
&\geq \frac{n(T'') + l(T'') - s(T'') + 1}{2} + 1 \\
&\geq \frac{n(T) - 2 + l(T) - s(T) + 1}{2} + 1 \\
&= \frac{n(T) + l(T) - s(T) + 1}{2}.
\end{aligned}$$

From the three cases above, the proof follows. □

Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence of trees $T_0, \ldots, T_k = T$, with $k \geq 0$ and $T_0 = P_5$. If $k \geq 1$, then for each subscript $i \in \{1, \ldots, k\}$, the tree T_i can be obtained from $T' = T_{i-1}$ by one of the next two operations:

- Operation O_1: Add a path P_1 and join it to a vertex $v \in \mathcal{S}(T')$.
- Operation O_2: Add a path P_2 and join one of its leaves to a vertex $v \in V(T') \setminus \mathcal{L}_s(T')$ which is in some $\gamma_2^{oi}(T')$-set.

Now, we proceed to prove that every tree in $\mathcal{T}$ achieves equality in the lower bound given in Theorem 5.

Lemma 1. *Let T be a tree of order $n(T) \geq 3$. If $T \in \mathcal{T}$, then $\gamma_2^{oi}(T) = \frac{n(T)+l(T)-s(T)+1}{2}$.*

Proof. Let T be a tree of order at least three. We proceed by induction on the number $r(T)$ of operations required to construct the tree T. If $r(T) = 0$, then $T = P_5$ and $\gamma_2^{oi}(T) = 3 = (n(T) + l(T) - s(T) + 1)/2$, as required. This establishes the base case. We now assume that $k \geq 1$ is an integer and that each tree $T^* \in \mathcal{T}$ with $r(T^*) < k$ satisfies that $\gamma_2^{oi}(T^*) = (n(T^*) + l(T^*) - s(T^*) + 1)/2$. Since $T \in \mathcal{T}$, it follows that T can be obtained from a tree $T' \in \mathcal{T}$ with $r(T') = k - 1$ by one of the Operations O_1 or O_2. Next, we consider the next two cases.

Case 1: T is obtained from T' by Operation O_1. In this case, T is obtained from T' by adding a vertex u_1 and the edge $u_1 v$, where $v \in \mathcal{S}(T')$. Observe that for any $\gamma_2^{oi}(T')$-set D', the set $D' \cup \{u_1\}$ is a 2OIDS of T. This implies that $\gamma_2^{oi}(T) \leq |D'| + 1 = \gamma_2^{oi}(T') + 1$. By using the lower bound given in Theorem 5, the induction hypothesis and the fact that $n(T) = n(T') + 1$, $l(T) = l(T') + 1$ and $s(T) = s(T')$, it follows that $(n(T) + l(T) - s(T) + 1)/2 \leq \gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1 = (n(T') + l(T') - s(T') + 1)/2 + 1 = (n(T) + l(T) - s(T) + 1)/2$. Hence, $\gamma_2^{oi}(T) = (n(T) + l(T) - s(T) + 1)/2$, as desired.

Case 2: T is obtained from T' by Operation O_2. In this case, T is obtained from T' by adding the path $u_1 u_2$ and the edge $u_1 v$, where $v \in D' \setminus \mathcal{L}_s(T')$ for some $\gamma_2^{oi}(T')$-set D'. Observe that $D' \cup \{u_2\}$ is a 2OIDS of T. This implies that $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. By using the lower bound given in Theorem 5, the induction hypothesis and the fact that $n(T) = n(T') + 2$ and $l(T) - s(T) = l(T') - s(T')$, we have that $(n(T) + l(T) - s(T) + 1)/2 \leq \gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1 = (n(T') + l(T') - s(T') + 1)/2 + 1 = (n(T) + l(T) - s(T) + 1)/2$. Hence, we have that $\gamma_2^{oi}(T) = (n(T) + l(T) - s(T) + 1)/2$, as desired. □

We next show that every tree T of order at least three satisfying equality $\gamma_2^{oi}(T) = \frac{n(T)+l(T)-s(T)+1}{2}$ belongs to the family $\mathcal{T}$.

Lemma 2. *Let T be a tree of order $n(T) \geq 3$. If $\gamma_2^{oi}(T) = \frac{n(T)+l(T)-s(T)+1}{2}$, then $T \in \mathcal{T}$.*

Proof. We proceed by induction on the order of a tree T, which satisfies that $\gamma_2^{oi}(T) = (n(T) + l(T) - s(T) + 1)/2$. If $n(T) \le 5$ then $T = P_5$, which belongs to $\mathcal{T}$. We assume that $n(T) > 5$ and that every tree T^* with $\gamma_2^{oi}(T^*) = (n(T^*) + l(T^*) - s(T^*) + 1)/2$ and $3 \le n(T^*) < n(T)$ satisfies that $T^* \in \mathcal{T}$. Now, we prove that $T \in \mathcal{T}$. For this, we root the tree T at a leaf v_1 belonging to a diametral path $v_1 \cdots v_d v_{d+1}$. We consider the following cases:

Case 1: $\deg(v_d) \ge 3$. Let us consider the subtree $T' = T - \{v_{d+1}\}$. Let D be a $\gamma_2^{oi}(T)$-set. As $\mathcal{L}(T) \subseteq D$ and $\overline{D}$ is an independent set of T, we have that $v_{d+1}, v_{d-1} \in D$, $v_d \in \overline{D}$ and $|(N(v_d) \cap D) \setminus \{v_{d+1}\}| \ge 2$. By the previous conditions, it is easy to deduce that $D \setminus \{v_{d+1}\}$ is a 2OIDS of T', which implies that $\gamma_2^{oi}(T') \le |D \setminus \{v_{d+1}\}| = \gamma_2^{oi}(T) - 1$. By using the lower bound given in Theorem 5 for the tree T', the hypothesis $\gamma_2^{oi}(T) = (n(T) + l(T) - s(T) + 1)/2$ and the fact that $n(T') = n(T) - 1$, $l(T') = l(T) - 1$ and $s(T') = s(T)$, it follows that

$$\frac{n(T') + l(T') - s(T') + 1}{2} \le \gamma_2^{oi}(T') \le \gamma_2^{oi}(T) - 1 = \frac{n(T') + l(T') - s(T') + 1}{2}.$$

Hence, we have that $\gamma_2^{oi}(T') = (n(T') + l(T') - s(T') + 1)/2$, and by the induction hypothesis, it follows that $T' \in \mathcal{T}$. Since $v_d \in \mathcal{S}(T')$, we have that T can be obtained from T' by Operation O_1, which implies that $T \in \mathcal{T}$, as desired.

Case 2: $\deg(v_d) = 2$. Let us consider the subtree $T'' = T - \{v_d, v_{d+1}\}$. Let D be a $\gamma_2^{oi}(T)$-set such that $v_{d-1} \in D$ and $v_d \in \overline{D}$ (such a set D exists because $\mathcal{L}(T) \subseteq D$, $\overline{D}$ is an independent set of T and $\deg(v_d) = 2$). Observe that $D'' = D \setminus \{v_{d+1}\}$ is a 2OIDS of T''. This implies that $\gamma_2^{oi}(T'') \le |D''| = \gamma_2^{oi}(T) - 1$. If $v_{d-1} \in \mathcal{L}_s(T'')$, then $l(T'') = l(T)$ and $s(T'') = s(T) - 1$, which implies that $\gamma_2^{oi}(T'') \le \gamma_2^{oi}(T) - 1 = (n(T) + l(T) - s(T) + 1)/2 - 1 < (n(T'') + l(T'') - s(T'') + 1)/2$, which contradicts the bound given in Theorem 5. Hence, $v_{d-1} \notin \mathcal{L}_s(T'')$, and as a consequence, it follows that $l(T'') = l(T) - 1$ and $s(T'') = s(T) - 1$. Therefore, by using the lower bound given in Theorem 5 for the tree T'', the hypothesis $\gamma_2^{oi}(T) = (n(T) + l(T) - s(T) + 1)/2$ and the previous equalities, we have that

$$\frac{n(T'') + l(T'') - s(T'') + 1}{2} \le \gamma_2^{oi}(T'') \le \gamma_2^{oi}(T) - 1 = \frac{n(T'') + l(T'') - s(T'') + 1}{2}.$$

Hence, we have that $\gamma_2^{oi}(T'') = (n(T'') + l(T'') - s(T'') + 1)/2$, and by the induction hypothesis, it follows that $T'' \in \mathcal{T}$. Moreover, it follows that $\gamma_2^{oi}(T'') = \gamma_2^{oi}(T) - 1$, which implies that D'' is a $\gamma_2^{oi}(T'')$-set containing vertex v_{d-1}. Since $v_{d-1} \notin \mathcal{L}_s(T'')$, we have that T can be obtained from tree T'' by Operation O_2, which implies that $T \in \mathcal{T}$, as desired. □

As an immediate consequence of Lemmas 1 and 2, we have the following characterization:

Theorem 6. *Let T be a tree of order at least three. Then, $\gamma_2^{oi}(T) = \frac{n(T)+l(T)-s(T)+1}{2}$ if and only if $T \in \mathcal{T}$.*

Let T^* be a tree obtained by subdividing the central edge of a double star exactly once. It is easy to see that the tree $T^* \in \mathcal{T}$ can only be obtained from the path P_5 by applying Operation O_1. On the other hand, observe that the path $P_7 \in \mathcal{T}$ can only be obtained from path P_5 by applying Operation O_2. Therefore, Operations O_1 and O_2 are required in the characterization above.

Finally, we show an interesting result, which is a consequence of Theorem 1. Observe that, by definition, it is easy to deduce that if G is a graph with $\delta(G) \ge 2$, then $\gamma_{\times 2}^{oi}(G) = \gamma_t^{oi}(G)$. However, characterizing the graphs G with $\delta(G) = 1$ that satisfy the above equality remains a problem to be solved. Next, we give a solution to this previous problem considering that G is a nontrivial tree.

Theorem 7. *Let T be a nontrivial tree. Then, $\gamma_{\times 2}^{oi}(T) = \gamma_t^{oi}(T)$ if and only if $T \cong P_2$.*

Proof. If $T \cong P_2$, then the equality $\gamma^{oi}_{\times 2}(T) = \gamma^{oi}_t(T)$ holds. From now on, we suppose that $T \ncong P_2$. We only need to prove that $\gamma^{oi}_t(T) < \gamma^{oi}_{\times 2}(T)$ (recall that $\gamma^{oi}_t(T) \le \gamma^{oi}_{\times 2}(T)$). If $T \cong P_3$, then we are done. Let us consider that $n(T) \ge 4$. From Theorem 1, and considering the upper bound given in (ii) and the lower bound given in (iii), we obtain the following inequality chain:

$$\gamma^{oi}_t(T) \le \frac{2n(T) - l(T) + s(T)}{3} < \frac{2n(T) + l(T) - s(T) + 2}{3} \le \gamma^{oi}_{\times 2}(T).$$

In particular, it follows that $\gamma^{oi}_t(T) < \gamma^{oi}_{\times 2}(T)$, as desired. Therefore, the proof is complete. □

3. Conclusions and Open Problems

This article is a contribution to the theory of domination in graphs. In particular, we studied three domination parameters in cactus graphs. Among the main contributions given in this article we emphasize the following.

- For each of the three domination parameters, we extended a well-known upper bound given for trees to the cactus graphs.
- We give a new lower bound for the 2-outer-independent domination number of a tree (in function on the order, the number of support vertices and the number of leaves), and we provide a constructive characterization of the trees that satisfy equality in that bound.
- We show that the double outer-independent domination number is greater than the total outer-independent domination number for any tree of order at least three.

To continue with this line of research, we propose some open problems, which we consider to be interesting:

1. Characterize the cactus graphs that satisfy the equalities in the upper bounds given in Theorems 2–4.
2. For each of the three outer-independent domination parameters, try to extend the lower bound given for trees to the cactus graphs.

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