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An extension of the 16-th Hilbert problem for continuous piecewise linear-quadratic centers separated by a non-regular line

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In these last decades there has been a big interest for studying piecewise diffeherential systems. This is mainly due to the fact that these differential systems allow to modelize many natural phenomena.

In order to describe the dynamics of a differential system we need to control its periodic orbits, and in special its limit cycles. In particular providing an upper bound for the maximum number of limit cycles that such differential systems can exhibit would be desirable, that is to solve the extended 16th Hilbert problem. In general this is an unsolved problem.

In this paper we give an upper bound for the maximum number of limit cycles that a class of continuous piecewise differential systems formed by an arbitrary linear center and an arbitrary quadratic center separated by a non-regular line can exhibit. So for this class of continuous piecewise differential systems we have solved the extended 16th Hilbert problem, and the upper bound found is seven. The question if this upper bound is sharp remains open.

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To solve the 16th Hilbert problem, that is, to give an upper bound for the maximum number of limit cycles that a given family of differential systems can exhibit, is in general an open problem. In this paper we obtain a solution of the 16th Hilbert problem for the class of continuous piecewise differential systems formed by a linear and a quadratic system, both having a center, whose continuity manifold is a non-regular line. The methodology includes the use of first integrals and the Chebyschev's Theory.

I. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In these last twenty years a big interest has appeared for understanding piecewise differential systems mainly due to their relevant applications in modeling many different natural phenomena, see for instance the books of Ref. 1, 2, and 31 and the survey of Ref. 26, and see the references cited in these books and in the survey.

In order to describe the dynamics of a differential system the periodic orbits play a main role (specially the limit cycles, i.e. the periodic orbits isolated in the set of all periodic orbits). Examples of relevant applications of the existence of limit cycles in the dynamics can be found in Ref. 2, 28, 29, and 32, ...

The existence and number of limit cycles of distinct classes of piecewise differential systems have been studied for many authors, see for instance Ref. 5–7, 9, 10, 12–16, 18–24, being the list not exhaustive.

In the present paper we consider continuous piecewise differential systems of the form

$$(\dot{x}, \dot{y}) = \begin{cases} \mathbf{F}_1(x,y) = (f_1(x,y), f_2(x,y)) \text{ if } (x,y) \in \mathcal{R}_{1\alpha}, \\ \mathbf{F}_2(x,y) = (g_1(x,y), g_2(x,y) \text{ if } (x,y) \in \mathcal{R}_{2\alpha}, \end{cases} \tag{1}$$

where the dot means derivative in the variable t and f_i, g_i for $i = 1, 2$ are, respectively, linear and quadratic polynomials. The regions $\mathcal{R}_{1\alpha}$ and $\mathcal{R}_{2\alpha}$ are

$$\mathcal{R}_{1\alpha} = \{(x,y) \in \mathbb{R}^2 : x = r\cos\theta, y = r\sin\theta, r \geq 0, 0 \leq \theta \leq \alpha\},$$
$$\mathcal{R}_{2\alpha} = \{(x,y) \in \mathbb{R}^2 : x = r\cos\theta, y = r\sin\theta, r \geq 0, \alpha \leq \theta \leq 2\pi\},$$

with $\alpha \in (0, \pi)$. We assume that $\mathbf{F}_1(x,y) = \mathbf{F}_2(x,y)$ if $(x,y) \in \mathcal{L}_\alpha = \mathcal{R}_{1\alpha} \cap \mathcal{R}_{2\alpha}$, i.e. the piecewise differential system is continuous on $\mathcal{L}_\alpha$.

We use a result in Ref. 16 which proves that any piecewise differential system of the form (1) by means of a linear transformation can be transformed into a piecewise differential system with $\alpha = \pi/2$. Thus without loss of generality in what follows we consider $\alpha = \pi/2$ and we shall write $\mathcal{R}_1 = \mathcal{R}_{1\pi/2}$, $\mathcal{R}_2 = \mathcal{R}_{2\pi/2}$ and $\mathcal{L} = \mathcal{L}_{\pi/2}$.

In this paper $(\dot{x}, \dot{y}) = \mathbf{F}_1(x, y)$ will be a linear differential system having a center, and $(\dot{x}, \dot{y}) = \mathbf{F}_2(x, y)$ will be a quadratic differential system having a center, simply called quadratic center.

In Ref. 23 it was proved that any linear differential system having a center can be written of the form

$$\dot{x} = -\alpha x - (\alpha^2 + \omega^2) y + \beta, \qquad \dot{y} = x + \alpha y + \gamma, \tag{2}$$

with $\omega > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha \neq 0$ and it has the first integral

$$H_1(x, y) = (x + \alpha y)^2 + 2(\gamma x - \beta y) + y^2 \omega^2. \tag{3}$$

So in this paper equation $(\dot{x}, \dot{y}) = \mathbf{F}_1(x, y)$ will be system (2) and for the equation $(\dot{x}, \dot{y}) = \mathbf{F}_2(x, y)$ we will consider the following generic quadratic polynomial differential system (simply quadratic system in what follows)

$$\dot{x} = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2, \qquad \dot{y} = d_0 + d_1 x + d_2 y + d_3 x^2 + d_4 xy + d_5 y^2, \tag{4}$$

and we will impose that it has a center.

Our objective is to provide an upper bound for the maximum number of limit cycles that these continuous piecewise differential systems can exhibit. The main result of the paper is the following one.

Theorem 1. *Seven is an upper bound for the maximum number of limit cycles that the continuous piecewise differential systems separated by the non-regular line $\mathcal{L}$ and formed by a linear differential center in the region $\mathcal{R}_1$ and a quadratic center in the region $\mathcal{R}_2$, or vice versa, can exhibit.*

The proof of Theorem 1 is given in section II. However it is unknown if this upper bound is reached.

II. PROOF OF THEOREM 1

In order to find an upper bound for the maximum number of limit cycles that the continuous piecewise differential system (2)-(4) can exhibit, we must impose two conditions to these systems. The first condition is about the continuity of the piecewise differential system along the separation line $\mathcal{L}$. The second condition is that the quadratic system (4) must have a center.

After a rescaling of the variables x, y and the time t we can consider $\alpha = 1$ in system (2). Now imposing the continuity of the piecewise differential system along the line $\mathcal{L}$, we obtain the following values for the coefficients of system (4)

$$c_0 = \beta,\ c_1 = -1,\ c_2 = -\omega^2 - 1,\ c_3 = 0,\ c_5 = 0, d_0 = \gamma,\ d_1 = 1,\ d_2 = 1,\ d_3 = 0,\ d_5 = 0.$$

Then the differential systems (2) and (4) become

$$\dot{x} = -x - (1 + \omega^2)y + \beta, \qquad \dot{y} = x + y + \gamma, \tag{5}$$

and

$$\dot{x} = \beta - x - (1 + \omega^2)y + c_4 xy, \qquad \dot{y} = \gamma + x + y + d_4 xy, \tag{6}$$

respectively, and they coincide on the line $\mathcal{L}$. Note that $c_4^2 + d_4^2 \neq 0$, otherwise the differential system (6) would not be a quadratic system.

Now we shall study the conditions in order that the quadratic system (6) has a center. We consider different cases.

Case 1*:* $c_4 + d_4 = 0$. In this case we shall prove that the quadratic system (6) never has a center. Computing the unique equilibrium point of system (6), we obtain that

$$(x_1, y_1) = \left(-\frac{\beta + \gamma(1 + \omega^2)}{d_4(\beta + \gamma) + \omega^2}, \frac{\beta + \gamma}{\omega^2} \right),$$

where we assume that $d_4(\beta + \gamma) + \omega^2 \neq 0$, otherwise system (6) has no equilibria for $c_4 + d_4 = 0$.

The equilibrium point (x_1, y_1) could be a center if the matrix of the linear part of (6) at that point has a zero trace and the discriminant of the characteristic polynomial of such a matrix is negative. These two conditions become

$$\frac{\beta+\gamma}{\omega^2}+\frac{\beta+\gamma(1+\omega^2)}{d_4(\beta+\gamma)+\omega^2}=0,$$

$$\frac{d_4^2(d_4(\beta+\gamma)^2+2(\beta+\gamma)\omega^2+\gamma\omega^4)^2}{\omega^4(d_4(\beta+\gamma)+\omega^2)^2}-4d_4(\beta+\gamma)-4\omega^2<0,$$

and so, we can obtain that

$$\beta=-\frac{d_4\gamma+(1-\sqrt{(1-d_4\gamma)})\omega^2}{d_4},\quad \gamma=\frac{1-k^2}{d_4}, \tag{7}$$

where we have introduced a new positive parameter $k\in\mathbb{R}$. Under these assumptions, the equilibrium point (x_1,y_1) could be a weak focus or a center. In order to prove that it is a center, we apply Theorem 2 given in the Appendix. To do that, we must write system (6) with the parameters β, γ given in (7) as in the form of Theorem 2. The first step, consists in applying the change of variables $X=x-x_1$ and $Y=y-y_1$, to move the equilibrium point to the origin of coordinates. After this change of variables, system (6) writes

$$\dot{x}=-kx-(k+\omega^2)y-d_4xy,\qquad \dot{y}=k(x+y)+d_4xy, \tag{8}$$

where we have renamed the new variables (X,Y) by (x,y).

Now if we write the real Jordan form of the linear part of system (8) at the origin, we get

$$J=\begin{pmatrix}0 & -\sqrt{k}\omega\\ \sqrt{k}\omega & 0\end{pmatrix},$$

and for this we do a new change of variables $(x,y)\to(u,v)$ given by

$$\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\frac{\sqrt{k}}{\omega} & -\frac{\sqrt{k}}{\omega}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Then, the quadratic system (8) becomes

$$\dot{u}=-v-\frac{d_4}{\sqrt{k}\omega}u^2-\frac{d_4}{k}uv,\qquad \dot{v}=u. \tag{9}$$

For system (9) the coefficients of Theorem 2 are $A=a=d=0$, $b=d_4/(\sqrt{k}\omega)$ and $C=d_4/k$. Then the conditions for having a center at the origin of system (9) cannot be fulfilled, and so the equilibrium is not a center.

Case 2*:* $c_4 + d_4(1+\omega^2) = 0$. In this case we also prove that the quadratic system (6) never has a center.

Computing the equilibrium points of system (6), we get that if $c_4 = -d_4(1+\omega^2)$, we have only one fixed point, namely

$$(x_1, y_1) = \left(-\frac{\beta+\gamma(1+\omega)^2}{\omega^2}, \frac{\beta+\gamma}{\omega^2 - d_4\left(\beta+\gamma(1+\omega^2)\right)} \right).$$

From now on we assume that $Q = \omega^2 - d_4\left(\beta+\gamma(1+\omega^2)\right) \neq 0$, otherwise there is no equilibrium point of system (6). We distinguish two different subcases.

Subcase 2.1*:* $\beta+\gamma(1+\omega^2) \neq 0$. A necessary condition for (x_1, y_1) to be a center is that the determinant Q of the matrix J of the linear part of system (6) at this point must be positive. Moreover, the trace of J must be zero, an so we obtain

$$d_4 = \frac{2(\beta+\gamma)\omega^2 + (\beta+2\gamma)\omega^4}{\left(\beta+\gamma(1+\omega^2)\right)^2}. \tag{10}$$

Then the discriminant of the characteristic polynomial of J is $\Delta = -4Q < 0$, and so the equilibrium point (x_1, y_1) is either a weak focus or a center. In order to check if it could be a center, we will apply Theorem 2 given in the appendix.

First we write system (6) in the normal form of Theorem 2. We start by applying the change of variables $X = x - x_1$, $Y = y - y_1$, so that the equilibrium point (x_1, y_1) is translated to the origin of coordinates (X, Y). Then we get the system

$$\begin{aligned}
\dot{x} &= \frac{1+\omega^2}{\left(\beta+\gamma(1+\omega^2)\right)^2}\left[(\beta+\gamma)(\beta+\gamma(1+\omega^2))\big(x+(1+\omega^2)y\big) - 2\omega^2(\beta+\gamma-(\beta+2\gamma)\omega^2)xy\right], \\
\dot{y} &= -\frac{1}{\left(\beta+\gamma(1+\omega^2)\right)^2(1+\omega^2)}\left[(\beta+\gamma(1+\omega^2))\big((\beta+\gamma(1-\omega^4))x+(\beta+\gamma)(1+\omega^2)^2 y\big)\right. \\
&\quad \left. -\big(\omega^2(1+\omega^2)(2(\beta+\gamma)+(\beta+2\gamma)\omega^2)\big)xy\right],
\end{aligned} \tag{11}$$

where we have renamed the new variables (X, Y) as the original ones (x, y).

Now we write the linear part of the system (11) at the origin in its real Jordan normal form, i.e.

$$J = \begin{pmatrix} 0 & -l \\ l & 0 \end{pmatrix}, \quad \text{where} \quad \ell = \frac{\sqrt{-(\beta+\gamma)}\,\omega\sqrt{1+\omega^2}}{\sqrt{\beta+\gamma(1+\omega^2)}}.$$

We must assume $\beta+\gamma \neq 0$, otherwise the eigenvalues of the equilibrium point at the origin of coordinates are both zero, and consequently the equilibrium point cannot be a center.

Now we do the changes of variables $(x, y) \to (u, v)$ given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{\ell}{\omega^2} & \dfrac{\ell(1+\omega^2)}{\omega^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then after a rescaling in the time variable (multiplying by the factor $1/\ell$), the quadratic system (11) becomes

$$\begin{aligned} \dot{u} =& -v + \frac{(\beta+\gamma)\omega\left(2(\beta+\gamma)+(\beta+2\gamma)\omega^2)\right)}{\sqrt{1+\omega^2}\left(-(\beta+\gamma)(\beta+\gamma(1+\omega)^2)\right)^{3/2}} u^2 + \frac{\omega^2\left(2(\beta+\gamma)+(\beta+2\gamma)\omega^2\right)}{(1+\omega^2)(\beta+\gamma)(\beta+\gamma(1+\omega^2))} uv, \\ \dot{v} =& u. \end{aligned} \tag{12}$$

For system (12) the coefficients A, a and d of Theorem 2 are $A = a = d = 0$. Then the conditions for having a center at the origin of system (12) are

$$\text{(i) b} = \text{C} = 0, \quad \text{(ii) C} = 0, \quad \text{(iii) b} = 0, \quad \text{(iv) C} = \text{b} = 0,$$

using the notations of Theorem 2. Then, we only have to study the cases when $b = 0$ and $C = 0$, because conditions (i) and (iv) are particular cases of (ii) and (iii).

However, the condition (ii) is not possible because $c_4 + d_4 \neq 0$ and being d_4 as in (10), we obtain that

$$c_4 + d_4 = \frac{\omega^4\left(2(\beta+\gamma)+(\beta+2\gamma)\omega^2\right)}{(\beta+\gamma(1+\omega^2)^2} \neq 0,$$

and so $C \neq 0$. For the same reasons, the condition (iii) $b = 0$, can not be possible in this case because $\beta + \gamma \neq 0$. We conclude that for the hypotheses in Subcase 2.1, system (6) never has a center.

Subcase 2.2*:* $\beta+\gamma(1+\omega^2) = 0$. Now the new obtained condition in order to have a matrix of the linear part of system (6) with positive determinant and zero trace is $\gamma = \beta = 0$. In this situation, the equilibrium (x_1, y_1) is at the origin and the discriminant is $\Delta = -4\omega^2 < 0$. Hence the origin is either a center or a weak focus. System (6) in this case becomes

$$\dot{x} = -x - (1+\omega^2)y - d_4(1+\omega^2)xy, \qquad \dot{y} = x + y + d_4 xy, \tag{13}$$

In order to apply Theorem 2 of the appendix to system (13) we shall write its linear part at the origin in its real Jordan normal form, i.e.

$$J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix},$$

and, we do the changes of variables $(x,y)\to(u,v)$ given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{\omega} & \dfrac{1+\omega^2}{\omega} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then after a rescaling in the time variable (multiplying by the factor $1/\omega$), the quadratic system (13) becomes

$$\dot{u} = -v + \frac{d_4}{\omega}u^2 - d_4 uv, \qquad \dot{v} = u. \tag{14}$$

For system (14) the coefficients A, a and d in Theorem 2 are all zero, and so the conditions for having a center are $b=0$ or $C=0$. However both conditions are not satisfied, because $d_4 \neq 0$. In conclusion there is no a center in Subcase 2.2.

Case 3: $c_4+d_4 \neq 0$, $c_4+d_4(1+\omega^2) \neq 0$ and $R = (-d_4\beta + c_4\gamma + \omega^2)^2 + 4(\beta + \gamma)\,(c_4+d_4(1+\omega^2)) \le 0$. Computing the equilibrium points of system (6) we see that if $R<0$ then all equilibria are complex, and so the quadratic system (6) cannot have a center. On the other hand, if $R=0$ the unique equilibrium point is the point

$$\left(\frac{d_4\beta - c_4\gamma + \omega^2}{2(c_4+d_4)}, -\frac{-d_4\beta + c_4\gamma + \omega^2}{2\,(c_4+d_4(1+\omega^2))}\right).$$

This equilibrium point comes from the collision of the two equilibrium points (x_1,y_1) and (x_2,y_2) of the Case 4, which are a saddle and a center. It is well-known that when a saddle and a center coalesce the resulting point is a saddle-node, and so in this case the quadratic system (6) cannot have a center.

Case 4: $c_4+d_4 \neq 0$, $c_4+d_4(1+\omega^2) \neq 0$ and $R = (-d_4\beta + c_4\gamma + \omega^2)^2 + 4(\beta + \gamma)\,(c_4+d_4(1+\omega^2)) > 0$. The equilibrium points of system (6) are

$$\begin{aligned} (x_1,y_1) &= \left(\frac{d_4\beta - c_4\gamma + \omega^2 + \sqrt{R}}{2(c_4+d_4)}, -\frac{-d_4\beta + c_4\gamma + \omega^2 + \sqrt{R}}{2\,(c_4+d_4(1+\omega^2))}\right), \\ (x_2,y_2) &= \left(\frac{d_4\beta - c_4\gamma + \omega^2 - \sqrt{R}}{2(c_4+d_4)}, -\frac{-d_4\beta + c_4\gamma + \omega^2 - \sqrt{R}}{2\,(c_4+d_4(1+\omega^2))}\right). \end{aligned}$$

The equilibrium point (x_1,y_1) is a saddle because the determinant of the matrix of the linear part of system (6) at it is negative. However the determinant of the matrix of the linear part of system (6) at the equilibrium point (x_2,y_2) is non-negative and so, it could be a center. A necessary condition in order that this equilibrium point be a center is that the discriminant of the characteristic polynomial of the matrix of the linear part of the system at that equilibrium be negative, i.e.

$$\begin{aligned}\Delta = {} & (c_4+d_4)^2(d_4^2(L^2+d_4^2\beta^2-2L(8+d_4\beta)) - 2c_4^3(L+d_4(\beta-\gamma))\gamma + c_4^4\gamma^2 + 2c_4d_4(-L(16+L)\\ & + d_4^2\beta^2 + d_4(L-d_4\beta)\gamma) + c_4^2(L^2+2L(-8+d_4\beta)+d_4^2(\beta^2-4\beta\gamma+\gamma^2))) + 2(c_4+d_4)(d_4^3(L^2\\ & + d_4\beta(1+d_4\beta) - L(17+2d_4\beta)) + c_4^4\gamma - c_4d_4^2(L(31+L)-d_4(\beta+d_4\beta^2-\gamma)\\ & + 2d_4(L-d_4\beta)\gamma) - c_4^2d_4(15L+d_4(\beta+\gamma+2d_4\beta\gamma-d_4\gamma^2)) - c_4^3(L+d_4(\beta-\gamma-d_4\gamma^2)))\omega^2\\ & + (c_4^4 + d_4^4(1+L^2+d_4\beta(4+d_4\beta)-2L(10+d_4\beta)) - 2c_4d_4^3(2(8L-d_4\beta)-d_4(2-L+d_4\beta)\gamma)\\ & - c_4^2d_4^2(2+12L+d_4\gamma(4-d_4\gamma)))\omega^4 - 2d_4^2(c_4^2-d_4^2(1-L+d_4\beta)+c_4d_4^2\gamma)\omega^6 + d_4^4\omega^8 < 0,\end{aligned} \tag{15}$$

where $R = L^2$. We introduce the new parameters $k \in \mathbb{R}\setminus\{0\}$ verifying $\Delta = -k^2$. Then the parameters β and γ in terms of k and L write

$$\begin{aligned}((c_4+d_4)^2+d_4^2\omega^2)^2\beta = {} & c_4(c_4+d_4)^2k^2 - L(5c_4^3+9c_4^2d_4+3c_4d_4^2-d_4^3-c_4^2d_4L)\\ & + (c_4+d_4)(c_4^2-d_4^2+c_4d_4k^2-5c_4d_4L+2d_4^2L)\omega^2 - d_4^2(c_4+2d_4-d_4L)\omega^4\\ & - d_4^3\omega^6 - 5d_4^2L)\omega^2 + d_4^3\omega^4 + ((c_4+d_4)(2c_4^2+4c_4d_4+2d_4^2+c_4^2L-c_4d_4L)\\ & + d_4(4c_4^2+8c_4d_4+4d_4^2-c_4d_4L)\omega^2 + 2d_4^2(c_4+d_4)\omega^4),\\ ((c_4+d_4)^2+d_4^2\omega^2)^2\gamma = {} & d_4(c_4+d_4)^2k^2 + L(c_4^3-3c_4^2d_4-9c_4d_4^2-5d_4^3+c_4d_4^2L)\\ & - (c_4+d_4)(c_4^2-d_4^2-d_4^2k^2+5d_4^2L)\omega^2 + d_4^3\omega^4 + \sqrt{4L-k^2}((c_4+d_4)(2c_4^2\\ & + 4c_4d_4+2d_4^2-c_4d_4L+d_4^2L) + d_4(2c_4^2+4c_4d_4+2d_4^2+d_4^2L)\omega^2).\end{aligned} \tag{16}$$

Another necessary condition in order that the equilibrium point (x_2, y_2) be a center is that the trace of the matrix of the linear part of system (6) at this equilibrium point be zero. Imposing this condition and using (16) we get that $L = k^2/4$. Using this relation the parameters β and γ become

$$\begin{aligned}\beta &= \frac{(k^2-4\omega^2)\left(-4c_4^3+c_4^2d_4(k^2-4)+4c_4d_4^2(1+\omega^2)+4d_4^3(1+\omega^2)^2\right)}{16\left((c_4+d_4)^2+d_4^2\omega^2\right)^2},\\ \gamma &= \frac{(k^2-4\omega^2)\left(4c_4^3+4c_4^2d_4+c_4d_4^2(k^2-4)-4d_4^3(1+\omega^2)\right)}{16\left((c_4+d_4)^2+d_4^2\omega^2\right)^2}.\end{aligned} \tag{17}$$

Under these assumptions the equilibrium point (x_2, y_2) could be a weak focus or a center. In order to ensure that it is a center we must apply Theorem 2 given in the appendix. For this we need to write system (6) with the parameters β, γ given in (17) in the normal form of Theorem 2. We first make the change of variables $X = x - x_2$, $Y = y - y_2$ so that the equilibrium point becomes at the origin origin of coordinates (X, Y). Doing so we get system

$$\begin{aligned}
\dot{x} = & \frac{1}{4((c_4+d_4)^2+d_4^2\omega^2)}\left[(-4c_4^2-4d_4^2(1+\omega^2)+c_4d_4\left(k^2-4(2+\omega^2)\right)x\right. \\
& \left.-\left(c_4^2(4+k^2)+8c_4d_4(1+\omega^2)+4d_4^2(1+\omega^2)^2\right)y\right]+c_4xy, \\
\dot{y} = & \frac{1}{4((c_4+d_4)^2+d_4^2\omega^2)}\left[\left(4(c_4+d_4)^2+d_4^2k^2\right)x+\left(4c_4^2+4d_4^2(1+\omega^2)+c_4d_4(8-k^2+4\omega^2)\right)y\right] \\
& +d_4xy.
\end{aligned} \tag{18}$$

where we have renamed the new variables (X, Y) as the old ones (x, y).

Now we write the linear part of the system (18) at the origin in its real Jordan normal form, i.e.

$$J = \begin{pmatrix} 0 & -\frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix},$$

for this we do the changes of variables $(x, y) \to (u, v)$ given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1+\dfrac{d_4\omega^2}{c_4+d_4} \\ -\dfrac{d_4k}{2(c_4+d_4)} & \dfrac{c_4k}{2(c_4+d_4)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then after a rescaling in the time variable (multiplying by the factor $2/k$), the quadratic system (18) becomes

$$\begin{aligned}
\dot{u} = & -v+\frac{2c_4d_4(c_4+d_4)}{k\left((c_4+d_4)^2+d_4^2\omega^2\right)}u^2+\frac{4(c_4+d_4)\left(c_4^2-d_4^4(1+\omega^2)\right)}{k^2\left((c_4+d_4)^2+d_4^2\omega^2\right)}uv-\frac{8(c_4+d_4)^2\left(c_4+d_4(1+\omega^2)\right)}{k^3\left((c_4+d_4)^2+d_4^2\omega^2\right)}v^2, \\
\dot{v} = & u.
\end{aligned} \tag{19}$$

For system (19) the coefficients A and a of Theorem 2 are $A = a = 0$. Then the conditions for having a center at the origin of system (19) are

$$\text{(i) } \mathrm{b}=\mathrm{C}=0, \quad \text{(ii) } \mathrm{C}=0, \quad \text{(iii) } \mathrm{b}+\mathrm{d}=0, \quad \text{(iv) } \mathrm{C}=3\mathrm{b}+5\mathrm{d}=\mathrm{bd}+2\mathrm{d}^2=0,$$

using the notations of Theorem 2. We see that conditions (i) and (iv) are particular cases of condition (ii), and so the necessary conditions are reduced to $b + d = 0$, or $C = 0$. We consider both cases separately.

Subcase 4.1*:* $b+d=0$. The condition $b+d=0$ is equivalent to

$$\frac{(c_4+d_4)\left(4c_4^2+4d_4^2(1+\omega^2)+c_4d_4(8-k^2+4\omega^2)\right)}{k^3\left((c_4+d_4)^2+d_4^2\omega^2\right)}=0. \tag{20}$$

Since $d_4(c_4+d_4)\neq 0$ (note that if $d_4=0$, then $c_4=0$ which is not possible), we obtain that

$$\omega^2=-\frac{4(c_4+d_4)^2-k^2c_4d_4}{4d_4(c_4+d_4)}>0,$$

otherwise we cannot have a center. We recall that $c_4\neq 0$, otherwise from (20) the above condition is never satisfied. System (19) with the obtained value of ω^2 has a center at the origin, and we can compute a first integral of it, see for instance Ref. 25 and 30.

Now we will show that the piecewise differential system (5)-(6) with the obtained values of β, γ and ω^2 has at most seven limit cycles. We first rewrite systems (5)-(6) and their first integrals in terms of the parameters k,c_4 and d_4. Then the lineal center becomes

$$\dot{x}=\frac{d_4k^2-4(c_4+d_4)}{4d_4(c_4+d_4)}-x+\frac{c_4\left(4(c_4+d_4)-d_4k^2\right)}{4d_4(c_4+d_4)}y,\qquad \dot{y}=\frac{1}{d_4}+x+y, \tag{21}$$

having the first integral becomes

$$H_1(x,y)=\frac{2}{d_4}x+\left(\frac{4(c_4+d_4)-k^2d_4}{2d_4(c_4+d_4)}\right)y+x^2+2xy-c_4\left(\frac{4(c_4+d_4)-k^2d_4}{4d_4(c_4+d_4)}\right)y^2; \tag{22}$$

and the quadratic center writes

$$\dot{x}=\frac{d_4k^2-4(c_4+d_4)}{4d_4(c_4+d_4)}-x+c_4\left(\frac{4(c_4+d_4)-k^2d_4}{4d_4(c_4+d_4)}\right)y+c_4xy,\qquad \dot{y}=\frac{1}{d_4}+x+y+d_4xy, \tag{23}$$

with the first integral

$$H_2(x,y)=e^{4d_4(c_4+d_4)(d_4x-c_4y)}\left(4(c_4+d_4)(1+d_4x)-k^2d_4\right)^{k^2d_4^2}(1+d_4y)^{4(c_4+d_4)^2}. \tag{24}$$

Consider a limit cycle that cuts the line $\mathcal{L}$ in the two points $(x,0)$ and $(0,y)$ with $x,y>0$. Then these points must satisfy the following two equations

$$e_1=H_1(x,0)-H_1(0,y)=0,\qquad e_2=H_2(x,0)-H_2(0,y)=0.$$

In order to study the solutions of the system $e_1=e_2=0$ we do the change of variables $(x,y)\to(X,Y)$ defined through

$$x=-\frac{1}{d_4}+\frac{1}{2}\sqrt{\frac{c_4(d_4k^2-4c_4-4d_4)}{d_4(c_4+d_4)}}(Y-X),\quad y=\frac{1}{c_4}+X+Y.$$

Note that under the assumptions of Subcase 4.1 the term $(c_4(d_4k^2-4c_4-4d_4))/(d_4(c_4+d_4)) > 0$. With this change of variables equation $e_1 = 0$ becomes

$$e_1 = (4(c_4+d_4)-d_4k^2)(4c_4(c_4+d_4)^2+c_4d_4^2k^2)^4(-4c_4^2-8c_4d_4-4d_4^2+d_4^2k^2+4c_4^2d_4(4(c_4+d_4)-d_4k^2)XY) = 0,$$

where we have removed the positive denominator $16d_4^3(c_4+d_4)^6$. Solving the equation $e_1 = 0$ with respect to the variable Y we get

$$Y = \frac{4(c_4+d_4)^2 - d_4^2k^2}{4c_4^2d_4(4(c_4+d_4)-d_4k^2)X}.$$

Now we introduce Y in $e_2 = 0$ and we can rewrite the resulting condition $e_2 = 0$ as $e_2 = a_0f_0 + a_1f_1 + a_2f_2$, being

$$f_0 = X^{\frac{4(c_4+d_4)^2}{k^2d_4^2}}, \qquad a_0 = 4(c_4+d_4) - d_4k^2,$$

$$f_1 = X^{1+\frac{4(c_4+d_4)^2}{k^2d_4^2}}, \qquad a_1 = 4d_4(c_4+d_4),$$

$$f_2 = e^{\frac{(c_4+d_4)((2(c_4+d_4^2)-d_4^2k^2)+4c_4d_4^2X^2(4(c_4+d_4)-d_4k^2))}{c_4d_4^2k^2X(4(c_4+d_4)-d_4k^2)}}\left(4c_4^2(4(c_4+d_4)-d_4k^2)X+4(c_4+d_4)^2-d_4^2k^2\right)^{\frac{4(c_4+d_4)^2}{d_4^2k^2}},$$

$$a_2 = -4^{-\frac{4(c_4+d_4)^2}{d_4^2k^2}}c_4^{-\frac{8(c_4+d_4)^2}{d_4^2k^2}}\left(4(c_4+d_4)-d_4k^2\right)^{1-\frac{4(c_4+d_4)^2}{d_4^2k^2}}.$$

The number of limit cycles of the piecewise differential system (21)-(23) depends on the number of positive solutions of the equation $e_2 = 0$. Therefore in order to obtain an upper bound for the number of positive solutions of $e_2 = 0$ we use the theory of Chebyschev, see Ref. 17 and 27 for details. Note that the functions $\{f_0, f_1, f_2\}$ are analytic in $(0, +\infty)$, and f_0 does not vanish in $(0, +\infty)$. The Wronskian $W[f_0, f_1]$ is

$$W[f_0, f_1] = X^{\frac{8(c_4+d_4)^2}{k^2d_4^2}} \neq 0 \quad \text{for } x \in (0, +\infty).$$

Moreover the Wronskian $W[f_0, f_1, f_2]$ is

$$W[f_0, f_1, f_2] = (c_4+d_4)e^{\frac{(c_4+d_4)((2(c_4+d_4^2)-d_4^2k^2)+4c_4d_4^2X^2(4(c_4+d_4)-d_4k^2))}{c_4d_4^2k^2X(4(c_4+d_4)-d_4k^2)}}X^{-4+\frac{8(c_4+d_4)^2k^2}{d_4^2}}$$
$$\left(4c_4^2(4(c_4+d_4)-d_4k^2)X+4(c_4+d_4)^2-d_4^2k^2\right)^{-2+\frac{4(c_4+d_4)^2}{d_4^2k^2}}P_6(X),$$

where

$$P_6(X) = C_0 + C_1X + C_2X^2 + C_3X^3 + C_4X^4 + C_5X^5 + C_6X^6,$$

with

$C_0 = (c_4 + d_4)(2c_4 + 2d_4 - d_4k)^4(2c_4 + 2d_4 + d_4k)^4$,

$C_1 = -2c_4d_4(2c_4 + 2d_4 - d_4k)^3(2c_4 + 2d_4 + d_4k)^3(4c_4 + 4d_4 - d_4k^2)(4c_4 + 4d_4 + d_4k^2)$,

$C_2 = -4c_4d_4^2(2c_4 + 2d_4 - d_4k)^2(2c_4 + 2d_4 + d_4k)^2(4c_4 + 4d_4 - d_4k^2)(-8c_4^3 - 8c_4^2d_4 + 8c_4d_4^2 + 8d_4^3 + 12c_4^3k^2 + 12c_4^2d_4k^2 - 2c_4d_4^2k^2 - 2d_4^3k^2 - 3c_4^2d_4k^4 + c_4d_4^2k^4)$,

$C_3 = -32c_4^2d_4^2(2c_4 + 2d_4 - d_4k)(2c_4 + 2d_4 + d_4k)(4c_4 + 4d_4 - d_4k^2)^2(4c_4^4 + 8c_4^3d_4 - 8c_4d_4^3 - 4d_4^4 - 4c_4^3d_4k^2 - 5c_4^2d_4^2k^2 + d_4^4k^2 + c_4^2d_4^2k^4)$,

$C_4 = 16c_4^2d_4^3(c_4 + d_4)(2c_4 + 2d_4 - d_4k)(2c_4 + 2d_4 + d_4k)(4c_4 + 4d_4 - d_4k^2)^2(32c_4^3 + 36c_4^2d_4 + 8c_4d_4^2 + 4d_4^3 - 8c_4^2d_4k^2 - d_4^3k^2)$,

$C_5 = 128c_4^4d_4^4(c_4 + d_4)(2c_4 + 2d_4 - d_4k)(2c_4 + 2d_4 + d_4k)(4c_4 + 4d_4 - d_4k^2)^3$,

$C_6 = 256c_4^6d_4^4(c_4 + d_4)(4c_4 + 4d_4 - d_4k^2)^4$.

Using the Descartes rule of signs with the help of the instruction Reduce of the algebraic manipulator Mathematica we obtain that the maximum number of positive solutions of the polynomial $P_6(X) = 0$ is four taking into account their multiplicities. If these four solutions are simple, then applying Theorem 1.2 of Ref. 27, with $n = 2$, $\nu_1 = \nu_0 = 0$ and all $\mu = 0$, we obtain that the maximum number of positive zeros of the Wronskian $W[f_0, f_1, f_2]$ is 7 (we recall that among the four possible positive real roots of $P_6 = 0$ there is also a possible simple positive real root of $4c_4^2(4(c_4 + d_4) - d_4k^2)X + 4(c_4 + d_4)^2 - d_4^2k^2 = 0$). If some of the roots of the polynomial $P_6(X)$ is not simple, then after a small perturbation it splits in simple roots, and we apply the previous argument. The proof of Theorem 1 is done in this case.

Subcase 4.2*:* $C = 0$. In this case it follows from system (19), that

$$C = -\frac{4(c_4 + d_4)\left(c_4^2 - d_4^2(1 + \omega^2)\right)}{k^2\left((c_4 + d_4)^2 + d_4^2\omega^2\right)} = 0$$

if and only if $c_4^2 = d_4^2(1 + \omega^2)$, (i.e. $\omega^2 = (c_4/d_4)^2 - 1$) because $c_4 + d_4 \neq 0$. Since the trace of the matrix of the linear part of system (6) at the equilibrium (x_2, y_2) must be zero, we obtain the following condition

$$\frac{d_4(d_4\beta - c_4\gamma + \omega^2 - \sqrt{R})}{c_4 + d_4} + \frac{c_4(d_4\beta - c_4\gamma - \omega^2 + \sqrt{R})}{c_4 + d_4(1 + \omega^2)} = 0;$$

which is equivalent to

$$\frac{d_4(d_4\beta - c_4\gamma)}{c_4 + d_4} = 0 \quad \text{that is} \quad d_4\beta - c_4\gamma = 0.$$

So if $\beta = c_4\gamma/d_4$, and the determinant of the linear part of system (6) at the equilibrium (x_2, y_2) is positive, i.e. $(c_4+d_4)^2 + 4c_4d_4^2\gamma > 0$ we have a center at (x_2, y_2) and so as in Subcase 4.1 we can compute a first integral of it.

Now we will show that the piecewise differential system (5)-(6) with the obtained values of β and ω has at most three limit cycles. We first rewrite systems (5)-(6) and their first integrals in terms of the parameters k,c_4 and d_4. Then the lineal center becomes

$$\dot{x} = \frac{c_4\gamma}{d_4} - x - \frac{c_4^2}{d_4^2}y, \qquad \dot{y} = \gamma + x + y, \tag{25}$$

having the first integral becomes

$$H_1(x,y) = 2\gamma x - \frac{2\gamma c_4}{d_4}y + x^2 + 2xy + \frac{c_4^2}{d_4^2}y^2; \tag{26}$$

and the quadratic center writes

$$\dot{x} = \frac{c_4\gamma}{d_4} - x - \frac{c_4^2}{d_4^2}y + c_4xy, \qquad \dot{y} = \gamma + x + y + d_4xy, \tag{27}$$

with the first integral

$$H_2(x,y) = (c_4 + d_4 - d_4^2\gamma - d_4^2x + c_4d_4y - d_4^3xy)e^{\frac{d_4(d_4x-c_4y)}{c_4+d_4}}. \tag{28}$$

Consider a limit cycle that cuts the line $\mathcal{L}$ in the two points $(x,0)$ and $(0,y)$ with $x, y > 0$. Then these points must satisfy the following two equations

$$e_1 = H_1(x,0) - H_1(0,y) = 0, \qquad e_2 = H_2(x,0) - H_2(0,y) = 0.$$

Solving the equation $e_1 = 0$ with respect to the variable x we get

$$x = -\frac{c_4}{d_4}y \quad \text{and} \quad x = -\frac{2d_4\gamma - c_4y}{d_4}.$$

Setting $x = -c_4y/d_4$ into e_2 we get $e_2 = 0$, so we have a continuum of solutions, and consequently they do not produce limit cycles.

On the other hand setting $x = -(2d_4\gamma - c_4y)/d_4$ into $e_2 = 0$ we get (after multiplying by $e^{c_4d_4y/(c_4+d_4)}$)

$$c_4 + d_4 - d_4^2\gamma + c_4d_4y - e^{-\frac{2\gamma d4^2}{c4+d4}}(c_4 + d_4 + d_4^2\gamma - c_4d_4y) = 0. \tag{29}$$

So equation $e_2 = 0$ can be written as $e_2 = c_0f_0 + c_1f_1 + c_2f_2 = 0$ with

$$\begin{aligned} f_0 &= 1, & c_0 &= c_4 + d_4 - \gamma d_4^2, \\ f_1 &= y\left(1 + e^{\frac{2d_4(c_4y-\gamma d_4)}{c_4+d_4}}\right), & c_1 &= c_4d_4, \\ f_2 &= e^{\frac{2d_4(c_4y-\gamma d_4)}{c_4+d_4}}, & c_2 &= -(c_4 + d_4 + \gamma d_4^2). \end{aligned}$$

The number of limit cycles of the piecewise differential system (25)-(27) depends on the number of positive solutions of the equation $e_2 = 0$. Therefore in order to obtain an upper bound for the number of positive solutions of $e_2 = 0$ we use the theory of Chebyschev, see Ref. 17. Note that the functions $\{f_0, f_1, f_2\}$ are analytic in $(0, +\infty)$. The Wronskian

$$W[f_0, f_1] = \frac{2c_4 d_4}{c_4 + d_4} e^{\frac{2d_4(c_4 y - d_4\gamma)}{c_4+d_4}} \neq 0 \quad \text{for } x \in (0, +\infty).$$

Moreover the Wronskian

$$W[f_0, f_1, f_2] = \frac{4c_4^2 d_4^2}{(c_4 + d_4)^2} e^{\frac{2d_4(c_4 y - d_4\gamma)}{c_4+d_4}} \left(e^{\frac{2d_4(c_4 y - d_4\gamma)}{c_4+d_4}} - 1 \right),$$

only vanishes on $y = d_4\gamma/c_4$. Applying Corollary 1.4 of Ref. 27, we obtain that the maximum number of positive zeros of the of equation (29) is three. Therefore in this case the continuous piecewise differential system can have at most three limit cycles. The proof of Theorem 1 is completed.

III. CONCLUSIONS

It is known that continuous piecewise differential systems formed by two linear-quadratic isochronous centers separated by a straight line do not have limit cycles Ref. 11, and if both systems are linear centers and the separation manifold is a non-regular line then they also do not have limit cycles, see Ref. 9.

In this paper, we have extended the study of giving an upper bound on the maximum number of limit cycles that a certain continuous system separated by a regular line can have. More precisely, we deal with the class of continuous piecewise linear-quadratic centers which are separated by a non-regular line. In fact, for this class of continuous differential systems we find that seven is an upper bound for the maximum number of limit cycles that they can exhibit, i.e. for these classes of differential systems we have solved the 16th Hilbert problem.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

IV. APPENDIX. THE KAPTEYN–BAUTIN THEOREM.

The following result gives us the characterization of quadratic centers.

Theorem 2 (Kapteyn–Bautin Theorem)**.** *A quadratic system that has a center at the origin can be written in the form*

$$\begin{aligned} \dot{x} &= -y - bx^2 - Cxy - dy^2, \\ \dot{y} &= x + ax^2 + Axy - ay^2. \end{aligned} \tag{A.1}$$

This system has a center at the origin if and only if, at least one of the following conditions holds

(i) $A - 2b = C + 2a = 0$,

(ii) $C = a = 0$,

(iii) $b + d = 0$,

(iv) $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$.

For a proof of this result see Theorem 8.15 of Ref. 4.

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