mathematics

Article

Developing Number Sense: An Approach to Initiate Algebraic Thinking in Primary Education

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Abstract: Traditionally, the teaching and learning of algebra has been addressed at the beginning of secondary education with a methodological approach that broke traumatically into a mathematical universe until now represented by numbers, with bad consequences. It is important, then, to find methodological alternatives that allow the parallel development of arithmetical and algebraic thinking from the first years of learning. This article begins with a review of a series of theoretical foundations that support a methodological proposal based on the use of specific manipulative materials that foster a deep knowledge of the decimal number system, while verbalizing and representing quantitative situations that underline numerical relationships and properties and patterns of numbers. Developing and illustrating this approach is the main purpose of this paper. The proposal has been implemented in a group of 25 pupils in the first year of primary school. Some observed milestones are presented and analyzed. In the light of the results, this well-planned early intervention contains key elements to initiate algebraic thinking through the development of number sense, naturally enhancing the translation of purely arithmetical situations into the symbolic language characteristic of algebraic thinking.

Keywords: number sense; algebra thinking; relational thinking; generalized arithmetic; tactical calculation

Citation: Adamuz-Povedano, N.; Fernández-Ahumada, E.; García-Pérez, M.T.; Montejo-Gámez, J. Developing Number Sense: An Approach to Initiate Algebraic Thinking in Primary Education. *Mathematics* **2021**, *9*, 518. https://doi.org/10.3390/math9050518

Received: 9 February 2021
Accepted: 25 February 2021
Published: 2 March 2021

1. Introduction

Nowadays, universal references to mathematics education as current policy frameworks of developed countries argue the importance of promoting experiences to develop the so-called number sense in early school activities [1]. Number sense is understood as a broad concept that includes a deep understanding of the decimal number system, with special emphasis on the relationships between numbers and operations, and the development, among others, of flexible mental calculation, numerical estimation and quantitative reasoning [2].

Moreover, one of the main objectives of mathematics at the beginning of secondary education focuses on becoming familiar with algebraic language and with the use of algebraic expressions to describe numerical relationships, which often present difficulties and even frustration for both students and teachers [3]. Although these difficulties are manifested in secondary education, their origin is long before, at the early levels of primary education, and due to various causes. Authors, such as those in [4], point out that in the 1960s it was already well-known that divergences between arithmetic and algebra can cause great difficulties in early algebra learning. [5] and [6] also stand out for the lack of understanding of arithmetic relationships. [7] indicates that there is a significant body of research that argues that children's misconceptions in algebra are due to a short-sighted approach to learning arithmetic in the early years. Furthermore, according to [8], there is a lack of work with properties of operations, regularities and patterns in elementary school. In addition, children are seldom placed in situations that lead to debate [9], to the analysis of expressions, to generalization or to the search for different solutions. To these causes, we should add the fact that classrooms, especially in the Spanish context, have neither concrete or well-structured manipulative resources to support verbalization and reasoning, nor a process of translating the experience to symbolic language [10].

If this trend is to be reversed, it seems logical to consider alternatives to the teaching–learning process of arithmetic in such a way as to foster algebraic thinking, in the sense that special emphasis is placed on the properties of operations, and generalizations are made when working with arithmetic [11]. A major body of early algebra research refers to this as generalized arithmetic [12], where two issues stand out: understanding the properties of operations and understanding the properties and relationships of numbers, being both much more important than getting the correct result [8]. The work presented here is aligned with this stream of research. This paper aims to develop and illustrate a didactic proposal, implemented in a classroom of first year of primary education, that develops algebraic thinking through an arithmetic that transcends the mere knowledge of numbers and operations to pay attention to a deep understanding of the number system, relationships and mathematical structures. In other words, this proposal addresses the development of number sense by considering "an arithmetic–algebraic thinking, a new approach which underpins the construction of a cognitive structure that links both types of thinking" [9]. The proposal is based on the use of manipulative materials, rigorously structured, that act both as numerical and arithmetic supports; and the activities are presented as problematic situations to promote the reasoning and analysis. With this approach, authors also aim to contribute to filling the gap in this field of mathematics education, where it has been acknowledged that more research is needed on how to introduce early algebraic thinking [13] or which tasks and activities are most suitable for its development [14].

1.1. Theoretical Foundation

Before the presentation of manipulative resources and activities related thereto, various aspects are discussed considered key to the development of algebraic thinking in early stages. The bridge they establish between arithmetic and algebra is also highlighted and linked to the decisions we made for our proposal.

1.1.1. Visualization, Manipulation and Number-Space Linking

Manipulative resources are key to perform very advantageous visualizations that allow us to know and materialize both numbers and operations. Numerous authors highlight the importance of visualization for the development of intuition, the configuration of internal representations and a better understanding of the mathematical concepts treated [15–19]. Montessori methodology asserts that the senses are the gateways to the mind, and that intellectual development requires sensory enrichment and hands-on activity [20,21]. We share this premise in our work with mathematics: the elaboration of abstract ideas is always based on the manipulation of materials. In all the activities we carry out, the material resource is the starting point, the support for the argumentation and the tool that facilitates the translation of the sensory-motor experience into symbolic language. Other authors add that the historical development of algebra has shown the importance of visualization as a fundamental tool to formulate arguments and algebraic formulas [18,22].

In addition to the visual properties of the resources, their sensory and motor properties should be highlighted, as they invite manipulation and movement. In this way, another way of teaching and learning mathematics emerges, as it is acknowledged that mathematical cognition is embodied and is closely connected to our sensorimotor functioning [23]. Reinforcing this idea, [24,25] show in several works an analysis of the algebraic thinking of children taking into account not only the use of language, but also the spatial descriptions and gestures they make. Moreover, this author states that the fact that letters are being used in certain contexts does not imply that algebra is being done, since algebraic thinking is basically a way of reflecting mathematically, and it can be done with other semiotic representations [24].

Moreover, the field of neuroscience shows us that the parietal circuit, critical for mathematical knowledge, is also involved in the representation of space, and that these two functions are intimately related [26,27]. This strong link between number and space in the brain suggests that the instructional methods that link number and space are powerful teaching tools [28]. This is the reason why our proposal is based on the use of manipulative materials that keep this relationship in mind, providing students with a numerical support for learning the decimal number system and an arithmetical support for operations.

1.1.2. Development of Relational Thinking

Thinking algebraically means going beyond the mechanical realization of calculations to focus attention on relational aspects. According to [29], relational thinking consists of knowing what is to be done with specific problems, and being in a position to relate these procedures with more general mathematical knowledge, such as the properties of the numbering system or the properties of operations. In the same vein, for [30] relational thinking consists of paying attention to the relationships and fundamental properties of operations, rather than to calculation procedures, which entails the use of the properties of numbers and operations to transform mathematical expressions into an easier calculation. If the teaching and learning of arithmetic is focused on emphasizing these relations, it will foster a more consistent learning needed to further address the formal introduction of algebra skills. This way of thinking should be engaged every day from the beginning, proposing activities in which the students have to connect knowledge and ideas. According to these same authors, the more sense we give to the teaching and learning of arithmetic, the closer the link with algebra will be.

Most of the studies that we find in the literature about students' difficulties with algebra are based on the lack of understanding of two fundamental algebraic concepts: equivalence and relational thinking [31]. Therefore, if we want to establish a solid base in algebraic thinking, we must propose activities that encourage this relational thinking, such as the creative use of operations when relating numbers or the relationship between consecutive numbers or transforming an expression into another equivalent that is easier to solve, among others.

1.1.3. Development of Flexible Quantitative Thinking

Flexible quantitative thinking, defined by [32] as "the ability to think and react to a quantitative situation in different ways", provides ease in the use of alternative strategies to the routines of school calculation and gives rise to original thought patterns in the context of arithmetic [33].

According to [34], flexible calculation strategies are desirable because they are based on the structure, properties and relationships between numbers. They take place, therefore, from learning with an understanding of mathematics.

Ref [35] indicates that the quantitative/conceptual approach also provides an early route to algebraic symbols as it represents general numerical relationships, instead of specific calculations. Students, by performing their calculations openly, focus on the nature of the calculation, regardless of the result. In this way they can adapt more easily to the use of expressions rather than calculated values. Furthermore, [36] observes that the point of writing flexible expressions is to show a chain of reasoning, a rationale that is useful in both arithmetic and algebraic thinking. Open expressions make it easier for students to reflect on the effects of changing a numerical value in a given situation and thus encourage a shift in focus from particular to general relations.

Ref [35] recognize that quantitative thinking does not develop easily or quickly. In fact, they consider that it deserves years of attention and development, both because it increases the probability of success in algebra and because it makes arithmetic and algebraic knowledge more meaningful and productive.

In this line, the activities and the manipulative resources we present are framed to favor a systematic and constant work of the flexible thinking, as will be seen below.

1.1.4. Understanding and Generalization of Patterns

An important part of the research related to early algebraic thinking is centered on the processes of generalization, with it being a fundamental process not only for early algebra but for any mathematical activity [37]. According to [38], generalization implies the search for some property invariant inside of a class of objects, is the search for the general as something recurrent or stable. [39], relating to the generalization of patterns, affirms "When students perform a pattern generalization, it basically involves mutually coordinating their perceptual and symbolic inferential abilities so that they are able to construct and justify a plausible and algebraically useful structure that could be conveyed in the form of a direct formula."

The finding of patterns and generalization are fundamental skills that we can develop from arithmetic and that will promote the subsequent formal study of algebra. Supporting this idea, [40] assert that the algebraization of arithmetic can promote a thought that supports algebra.

Finding the regularities and changes that occur in the numerical sequence is a priority task, since it will lead students to check that there is a recurrence in the system and, by doing so, they facilitate tasks of anticipation, representation of sections of major numbers and mental calculations. On this "symbolic certainty", that is, on the full understanding of the numerical order, they can construct arithmetic patterns and extend them safely to any process.

The use of the resources that we show below facilitates this work; for example, we will see how they make evident the recurrence or the pattern of the distances between numbers.

1.1.5. Non-Algorithmic Calculation

In traditional algorithms, children cannot dump what they learn from numbers and operations, they can only follow automated and meaningless steps, since the transformations they perform are hidden behind the "elegant notation of the algorithm" [30]. Teaching based on comprehension, reasoning and acquisition of skills with numbers cannot use traditional algorithms, but should evolve to creative resolution procedures, transparent algorithms that show the "know-how" of the student and let the teachers see the mental process that the student has followed.

Our proposal for the written calculation faithfully connects the manipulative resources we use and we refer to it as tactical calculation [41]. It is a procedure that takes the initial instruction and rewrites it following a plan of action, a tactic that reflects the student's numerical sense and relational thinking. The way to make decision-making explicit is extensive, that is, the plan is specified as the numbers and signs are linked. This planning is never casual or improvised, the daily work in the classroom leads the students to know and apply effective tactics that facilitate the work of the brain to execute the chain of calculations. We believe that, in addition to the obvious advantages, this modality provides children with skills in arithmetic structures that will undoubtedly benefit the algebraic modes of graphic expression.

2. Materials and Methods

The didactic proposal presented in this paper is aimed at pupils in the first year of primary education. All the examples shown correspond to the productions of a group of 25 pupils of a public school (CEIP Al-Andalus) in a city of southern Spain.

In this section, we present the teaching materials and some transversal issues on which the proposal is based.

2.1. Teaching Materials

This proposal is based on the use of the following manipulative materials, rigorously structured:

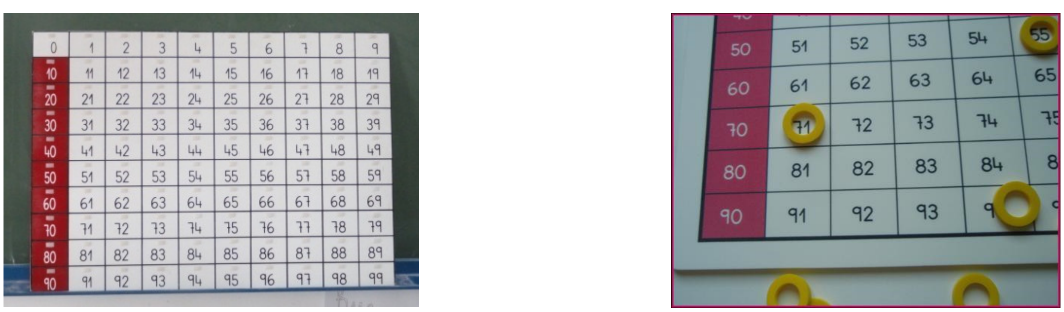
- Number line/tape (Figure 1): It is a tape which facilitates the appropriation of numbers as a linearly ordered, continuous and expandable sequence. It starts with 0, number used as starting point, absence of accounting elements or total loss, and end with 100, as a gateway to the numbers with hundreds that will be studied later on.

Figure 1. Detail of the number line.

The tape shows the pattern that generates numbers and distances. The exact tens (10, 20, 30...) have a red background, which makes it easier to establish clear references to numerical distances. The physical distance between consecutive exact tens is always the same, which makes it possible to establish a correspondence between physical and numerical distance ("from 10 to 20 there is the same as from 30 to 40"), and even to extend it to distances between any numbers ("from 12 to 22 there is the same as from 32 to 42"). Thus, the tape provides a spatial visualization that makes it easier to locate special places (the exact tens in red), to compare and to estimate distances. Precisely those places in red appear as strategic enclaves for calculation.

- Numerical panel: it presents the numbers from zero to ninety-nine by families, enabling new possibilities of analysis and relation.

There are two versions, the large panel is used collectively in the classroom and the small panel will be used individually (Figure 2).



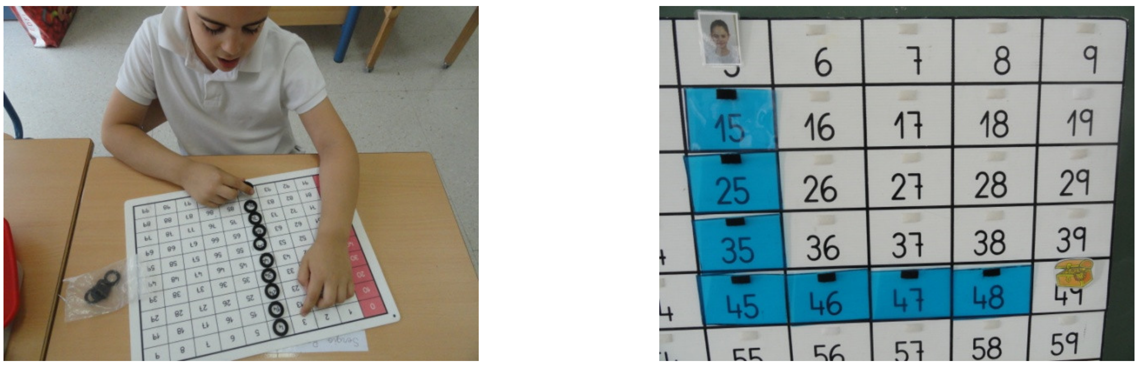
(**a**) (**b**)

Figure 2. (**a**) Collective numerical panel; (**b**) Personal numerical panel.

The exact tens have also a red background, in order to facilitate their location and to point them out as key elements in our numbering system. Likewise, the role of the first row should be highlighted as the first numbers studied, with which we compare the rest of the number families, as well as appreciating the regularities that this arrangement provides. The activities with the panels complement those on the tape and with other classroom resources, providing pupils with greater flexibility in reasoning about numbers and their properties.

As teachers get to know and use it in the classroom, it becomes easier for students to discover regularities and define the relationship between the elements that belong to the same row or to the same column.

In Figure 3 the use of the panel is exemplified as a map of numbers; in Figure 3a the student counts 10 by 10 in the panel, marking all the numbers of the same column. In Figure 3b the route that a student followed to go from 5 to 49 is shown.



(**a**) (**b**)

Figure 3. (**a**) Counting in tenths; (**b**) Route from 5 to 49.

These resources provide a numerical mapping, that is, a geometric representation onto which students can locate positions and plot personal calculation routes with total certainty. When the student internalizes these maps and thinks in a specific number, adding to its location, he/she also activates what he/she knows of that number, that is, the experience that he/she has accumulated about its uses, its size, its decimal structure, the possibilities of decomposition, distances to strategic locations, etc.

- Numbering box (Figure 4): it is a box with the separation of units, tens and hundreds. Additionally, numerous plastic sticks and red and green rubber bands are used. The whole resource is essential for understanding and working with our decimal number system. Through the use of the sticks and rubber bands, teachers will introduce students to the characteristics and rules of the system. They will be able to see equivalences and will manage to differentiate between different parts of the number without losing the idea of totality.

Figure 4. Numbering box.

The numbering box gives meaning to the written symbols, i.e., how quantities are represented symbolically. By connecting the representation in the box with the numbers on the tape and the panel, children will perceive that the position of the number is relevant, as each space in the box holds very different units.

Students will be able to carry out the first grouping in tens and the visualization of place value as well as other more complex processes such as the differentiated handling of unit orders or the flexible decomposition outside the box. In Figure 5 a work of flexible decomposition of the quantity ninety-nine is illustrated: on one side, the student has separated forty-one and, on the other side, fifty-eight. She writes on a sheet the arithmetic expression that underlies each breakdown. This is a convenient decomposition for a calculation, different from the canonical decomposition provided by the numbering box.

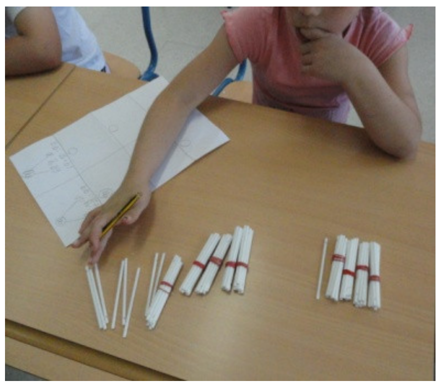


Figure 5. Flexible breakdown.

In terms of calculation, the numbering box is directly linked to the decomposition strategies and facilitates the graphical transcription resulting from the manipulation of quantities.

2.2. Transversal Issues

At the same time, we consider some fundamental aspects, which we could call transversal aspects, present in the context of instruction, enriching it and contributing with singularity, namely: the emotional climate and the communicative approach of languages. The first feature that the learning environment should have is that it is exciting. The research that comes from neuroscience assures that emotions are powerful allies in the instructional process, and positively or negatively affect critical elements such as attention, memory, initiative, self-esteem and performance [42]. Therefore, one of the great challenges that teachers face when teaching mathematics to their students is to provide a motivating environment that gives security and encourages participation, that invites reflection and that makes daily work a rewarding experience. The second aspect that we have labelled as transversal is the verbal expression that must accompany all mathematical activity, and also the interpretation and translation into the natural language of expressions with symbols and signs. The teacher has a crucial role in this complex process of coding and decoding, but meanings have to be constructed collectively, with the interaction of the whole group. [43] ensures that "if we consider mathematics as a language, communicative competence becomes an important issue, and meaningful communication a fundamental concern".

3. Results

Here we present the didactic proposal with a selection of activities, in which we emphasize aspects that connect arithmetic with algebraic thinking, that is, analysis of relationships between quantities, representation of relations, construction, interpretation and transformation of structures, patterns and generalization.

3.1. Relate a Number with the Previous ten and the Next Ten

This is a key activity for the mental calculation that teachers must repeat every time they expand the knowledge of a new family of numbers. By linking each number with the previous and the next ten, in addition to facilitating the calculations, important processes will be launched: connect the verbal expression of the reasoning and calculations with the corresponding written expressions, retake what has been learned about the basic combinations of ten to generalize it to situations in which multiples of ten are involved and show the inverse relationship between addition and subtraction.

Suppose the section between fifty and sixty is being studied. The teacher points out the number fifty-one with a clip on the number tape (Figure 6):

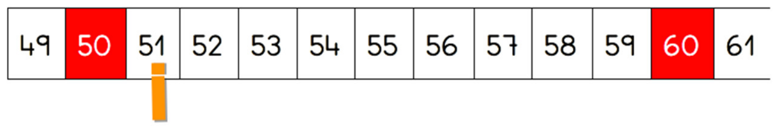


Figure 6. Detail of number tape.

Teacher: How can we relate this number to fifty? How can we go from one to the other with operations?

Student: If we give one to fifty, we reach fifty-one.

Student: And if we take one from fifty-one, we go back to fifty.

In this way, using the tape number as visual support, the teacher takes the group to observation, planning and verbal resolution. The teacher moves the clip from one number to the other to demonstrate what they say and then invite them to use the arithmetic structures as a means of expression.

$$50 + 1 = 51$$
$$51 - 1 = 50$$

Teacher: And with the next ten, how can we relate it to the number fifty-one?

Student: If we add nine, we get to sixty.

Student: And if we take away nine, we return to fifty-one.

The teacher verifies on the number tape and writes the new operations next to the previous ones. She/he reiterates making the appropriate checks and ending with the written expression:

$$50 + 1 = 51 \qquad 51 + 9 = 60$$
$$51 - 1 = 50 \qquad 60 - 9 = 51$$

It is common that many children anticipate what the teacher says and do not need to verify it. They know that they have seen these relationships before when studying other sections: it is the pattern of the "arithmetic of ten", which extends exactly to all families.

It is necessary to recognize the important role played by the tape as a precursor resource of relational thinking and generalization. With the practice of activities, such as the one shown, the relationship of any number with the ten that precede or follow it will become an automatic response. Thus, the student can focus on more complex aspects of the calculation, such as the choice of strategy.

Below, students' productions working on relating a number to the next ten are illustrated, based on the search for complements to ten of the first nine numbers. With these first exercises, the students begin in the layout and management of the Empty Number Line (ENL). Above it is represented the departure number, the arrival number, the direction of the movement and finally the operation that explains it. In Figure 7 we show some examples in which the instruction for getting number 10 has been given.

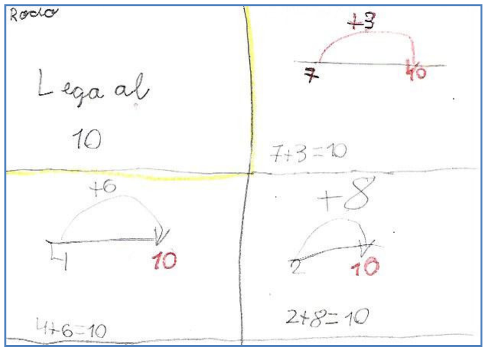


Figure 7. Rocío gets 10.

Later on, as other numerical sections are known, the activity is repeated, expanding it to relate to the following ten (Figure 8):

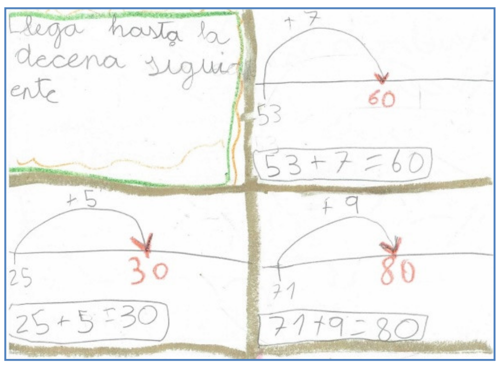


Figure 8. Get next ten.

At the same time, it is necessary to propose the return to the previous ten. When transcribing the ENL scheme to operation, students put the number from which they started the movement, its direction (-sign) and magnitude and the arrival number (Figure 9):

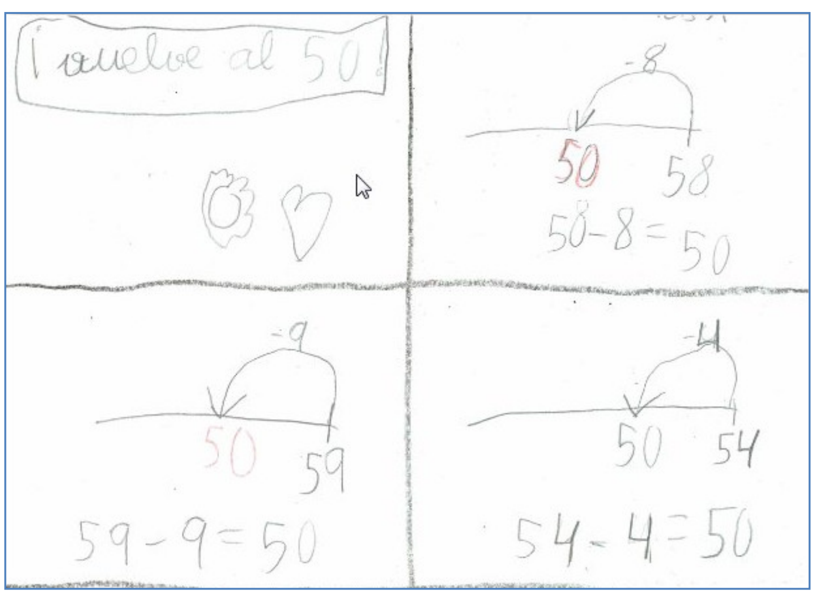


Figure 9. Return to the previous ten.

3.2. Verify and Generalize Numerical and Arithmetic Patterns

The understanding of the numerical pattern is an essential meta-knowledge in work with numbering and calculation that is difficult to assimilate by the students, since it requires a great level of abstraction. Teachers must facilitate access using resources that allow students to visualize series of numbers and develop global and local observation activities to detect regularities and changes. The tape and the panel are very useful in this process, because they show ordered sequences in which discover and verify both the recurrence of symbols and the periodicity in the changes.

The teacher points out the first nine natural numbers on the tape, which we use to count (Figure 10):

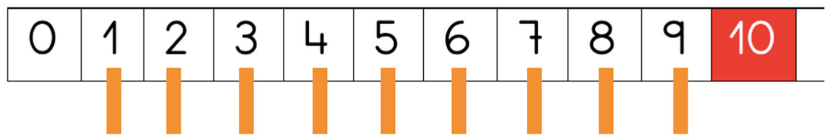


Figure 10. The first nine natural number highlighted.

We look at the complete number tape and reproduce this pattern to the highlighted numbers with a red background. We check that again happens with 1, 2, 3, 4, 5, 6, 7, 8 and 9 accompanied by zero, that is: 10, 20, 30, 40, 50, 60, 70, 80 and 90. We now look at the space between every two tens, and we verify the recurrence in the second digit (Figure 11).

Figure 11. Other family number highlighted.

The teacher then turns the attention to the changes of family, to identify the common pattern in them: 8, 9, 10, 11, 12... 18, 19, 20, 21, 22... 28, 29, 30, 31, 32... 3 8, 39, 40, 41, 42...

All this work will lead students to a conviction in the symbolic plane that will be fundamental for the mental journey through the sequence. The calculation will thus become a safe and predictable advance or setback, a "trip" between fixed positions that they can plan with increasing efficiency as they learn to manage the numbers.

Although the linear format in the number tape is the best recourse to contribute to building a mental line, teachers must complement the display of numbers in the two-dimensional plane which show the panel. This configuration by rows and columns shows aspects of the numerical pattern that are very enriching as well as profitable for the calculation.

In the first row we have zero (absence of quantity) next to the first nine numbers, with which we learn to count. In the first column we have the exact tens from ten to ninety. The first row and the red column are the fundamental lines, and they are the generating axes of all other positions. In this rigorous map, each number has a place and coordinates: the first corresponds to the family to which belongs, and the second to the number that heads the column. So, when studying each family, students see that the first digit is common and that the second one changes following the pattern marked by the first row. When children understand the exact rhythm of the numbers and perceive the spatial logic of the panel, they are able to imagine not only a specific location, but also the horizontal and vertical routes that lead from one number to another. So, it will be much easier to build other patterns linked to the calculation: the arithmetic patterns.

The construction of arithmetic patterns is developed with our resources at different levels of complexity, moving from simple processes to others that require more knowledge. Of note is the beneficial visual reinforcement that both, panel (Figure 12) and number tape (Figure 13), lend in activities such as the following:

Automation of operations of type T + U (exact tens plus units), for example:

$$10 + 4 = 14$$
$$20 + 4 = 24$$
$$30 + 4 = 34$$
$$40 + 4 = 44$$

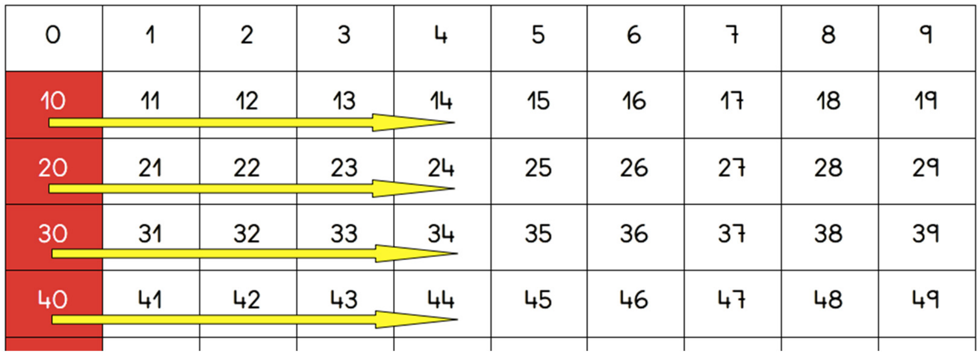


Figure 12. Panel with some operation represented.

Extension of the arithmetic of ten to operations of type T + U and T − U (addition and subtraction from exact tens), for example:

$$10 - 3 = 7$$
$$20 - 3 = 17$$
$$30 - 3 = 27$$

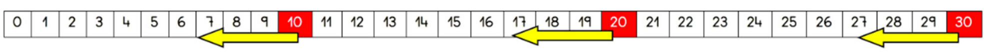


Figure 13. Number tape with some operations represented.

Acquisition of simple calculation strategies: the difference of ten between the same column of the panel creates a kind of "geometric memory" that can be very useful to "think" operations of type TU ± 9, TU ± 19, etc. (Figure 14).

38 + 19 = 38 + 20 − 1 = 57 (Green path)

93 − 39 = 93 − 40 + 1 = 54 (Blue path)

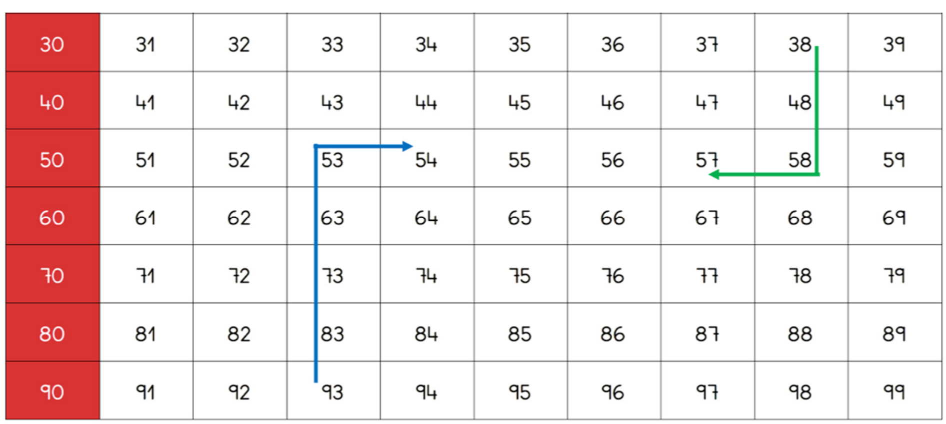


Figure 14. Detail of two path on the panel.

Acquisition of more complex calculation strategies: when the group reaches a good level of mastery, the panel format can be extended to the hundreds to work with numbers greater than one hundred and to implement strategies of partial approximation (in sums) or partial withdrawal (in subtraction). At first, it is necessary for children to investigate and make experimental scores by going from one table to another, exploring the new territory. Later, the teacher can guide them towards a very effective tactic: the tactic of the first row. Now, the teacher will work with a representation of the folio-sized panel enlarged to the first hundred. In the first place, they will remember the complements to ten that they have used several times to solve calculations and they will apply them extending to the exact tens. All together they go down the red column and review at the oral and written level all the movements going to the hundred and back to the number. It will be very easy because it deals with numbers with zeros and also because it is a sufficiently worked content about which they already have symbolic certainty, for example:

> Teacher: If I add ten to ninety, I'll reach one hundred, and a hundred minus ninety returns to ten... how do we express this relationship between ten and one hundred?

$$10 + 90 = 100 \qquad 100 - 90 = 10\ldots$$

Once all the combinations are remembered, the teacher calls attention to the row of a hundred, the "first line". Now, they are going to put the row of ten in relation to the row of the hundred to quantify again the distances and reflect on it (Figure 15):

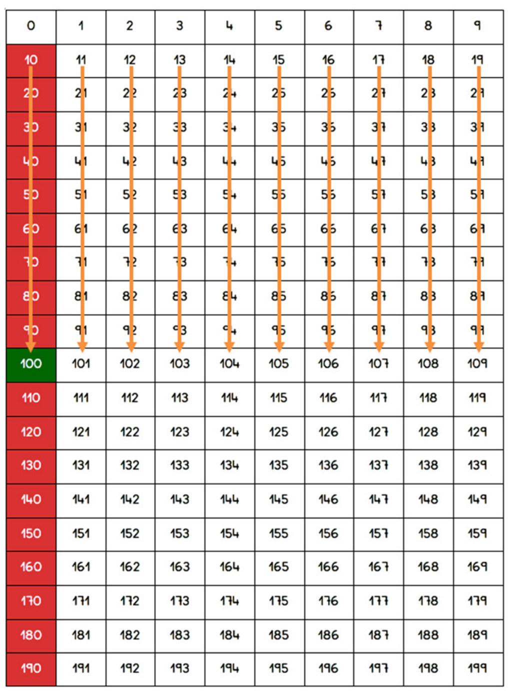


Figure 15. Getting the first line.

It is verified that ninety, which was the distance of ten to one hundred, now becomes a constant to travel in a straight line from one row to the other. From the verbal expression they turn to the written one:

$$11 + 90 = 101 \qquad 101 - 90 = 11$$
$$12 + 90 = 102 \qquad 102 - 90 = 12$$
$$13 + 90 = 103 \qquad 103 - 90 = 13$$

So, they are with all the other numbers of the family.

Step by step, with daily work in the classroom, the teacher will develop all the expressions that connect the rest of the exact tens with the first row of the first hundred. Later, when studying the other hundreds, they will repeat this work to confirm that the same arithmetic pattern is repeated. When the process is mastered, they can start the application of the tactics to the calculation. To work out 76 + 53, for example, they do a first planning: they add 30 to reach the first row, they keep going down with +20 and they advance three with +3:

$$76 + 30 + 20 + 3 =$$

The second phase is the resolution, and it is done by following the established route (Figure 16):

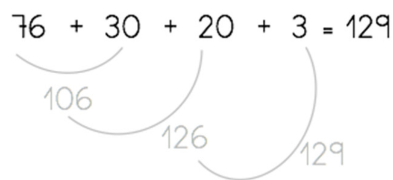


Figure 16. Route followed for the resolution.

In Figure 17, a work of extension of arithmetic patterns is illustrated. Using the number tape, the subtraction of units from sixty has worked, relating it to the arithmetic of ten. From there, the subtraction of numbers with a ten has been introduced. The numerical pattern of the tape allows for visualizing these two operations and becoming aware of the generalizable aspects.

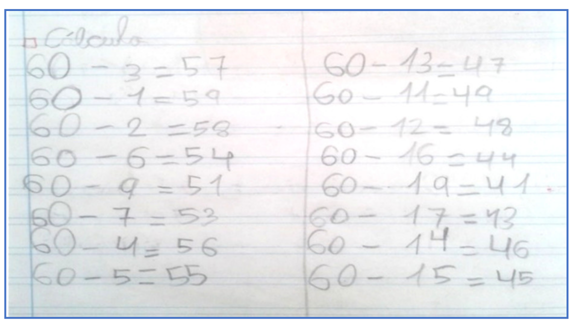


Figure 17. Example of arithmetic patterns.

In the activity in Figure 18 students have started with the visualization of basic subtractions (minuend between one and nine) that were represented in the tape. Then, these subtractions were extended to other families and the calculation was expressed in writing on the paper. In each quadrant of the activity, the minuend was increasing ten and the subtrahend remained constant. The aim was to anticipate the result, that is, to understand the arithmetic pattern that emerges from these subtractions.

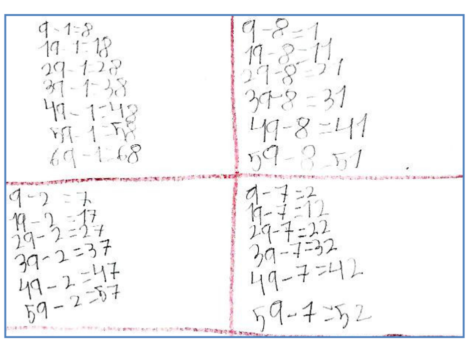


Figure 18. Example of arithmetic pattern in subtraction.

In Figure 19, we show how the student even draws the movement that would be made in the panel as a result of that calculation: to add 9, first goes down one box adding ten; then goes one back to the left and subtracts one.

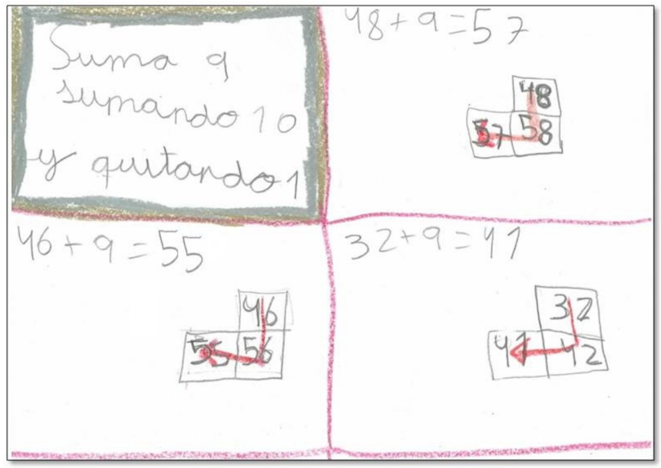


Figure 19. Operation and its representation of movement on the panel.

3.3. Know and Understand the Decimal Number System

With the numbering box, students will be able to materialize abstract contents such as the concept of units and the positional value. As we mentioned before, it is a totally manipulative resource with which students can build the numbers one by one and get to understand their structure and size. The box should be used in connection with the number tape or the panel to achieve a complete knowledge of the cardinal and ordinal aspects of the number. Therefore, very often the symbolic representation of the number tape or the panel will be connected with the concrete representation that the box offers:

With the box, students can give meaning to the symbols: the ten (10) has a ten (1) and no loose unit (0). They can also verify the positional value: in the tens there are ten sticks grouped (first value is ten), one of the units is worth one (Figure 20).

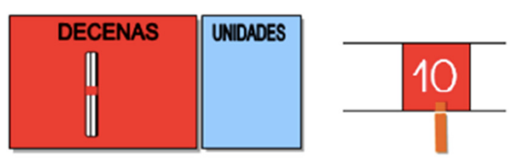


Figure 20. Number ten in box and number tape.

When placing the quantities in the box, they are presented separately in tens and units. Step by step teachers must guide the reflection so that students do not pigeonhole themselves in this visualization and learn to see all the units simultaneously; that is, they see not only the quantity that exists in the place of the units, but also the units that are grouped in the tens. The rigor with which teachers use the language will help a lot in this process:

Teacher: As we can see, fifty-two is five tens and two units, but we can also say that it is fifty units and two more units. How can we express this number with a sum?

Student: We can express it like this: 52 = 50 + 2.

The mastery of these processes of composition and decomposition (breakdown) of the number ensures the basis to reflect on the numbers in a way that favors and speeds up mental calculation. In addition and subtraction operations, the differentiated handling that students can make of units and tens facilitates and gives meaning to the translation process to symbolic notation.

When the numbers are mastered until ninety-nine, the box widens its possibilities until nine hundred and ninety-nine by adding another space where the hundreds will be housed. This opens enormous possibilities of work to know and operate with the numbers that make up this new numerical section.

Furthermore, the foundation and deep understanding of a large number of strategies and tactics that are applied to the calculation is found in the work with the numbering box, because in it students work with real quantities, not with symbols. We could say that the box is the resource that provides the real certainty that enables to transfer the work to abstract entities. When students add or subtract with the box, they, by themselves, place the quantities and execute the transformations, besides which the symbolic transcription corresponds literally with what they have seen and manipulated previously. Therefore, it is a great ally in the development of flexible quantitative thinking. Let us see how:

Teacher: I suggest you do the operation 32 − 17 with the box.

The student places the amount (Figure 21) and plans the subtraction by expressing it orally.

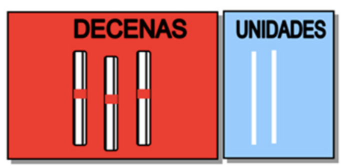


Figure 21. Number 32 represented in the box.

> Student: I have to take seventeen. First, I will remove the two units (Figure 22a), then a ten, which are ten (Figure 22b), then I will remove a rubber band to remove five units and put the rest of the units in place (Figure 22c).

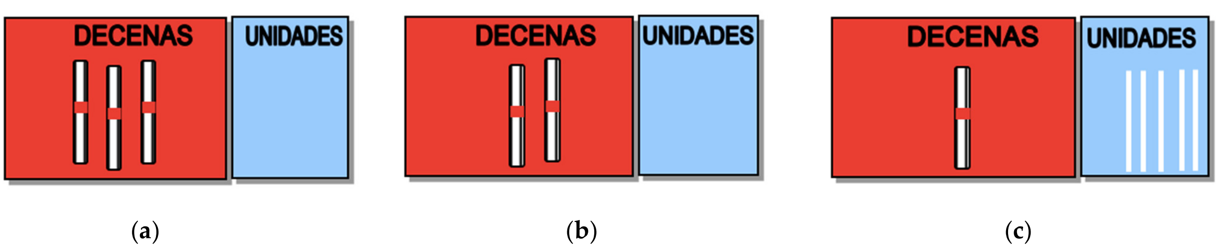


Figure 22. Process to take away 17 from 32: firstly, the student removes 2 units (**a**), secondly, one ten (**b**) and finally 5 units (**c**) by removing a rubber band of one ten.

Outside the box they will have ten and seven units and inside ten and five units. The translation to operation will be literal:

$$32 - 2 - 10 - 5 = 15$$

Later, when the experience is well internalized, they can resolve based on mental images.

Below there are examples of this work. The box provides a great support to understanding the operations and facilitate the work of translating them into numbers and signs.

In Figure 23 we can see how a student graphically creates a static addition proposal, that is, he/she is asked to put together the quantities of two boxes:

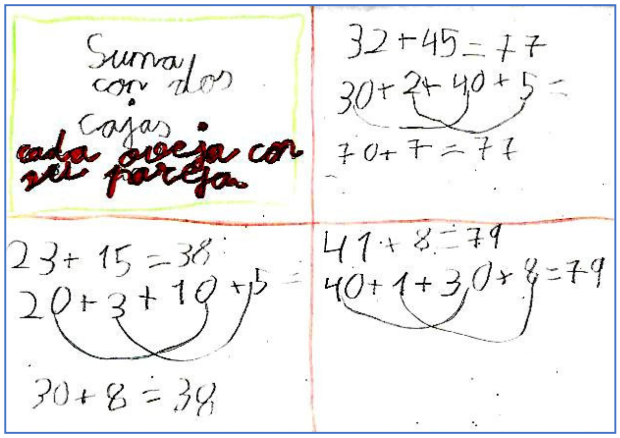


Figure 23. Addition with two boxes.

The work shown in Figure 24 follows a dynamic model for operations. The tactic is to go "little by little", controlling the increase or detraction of the amounts:

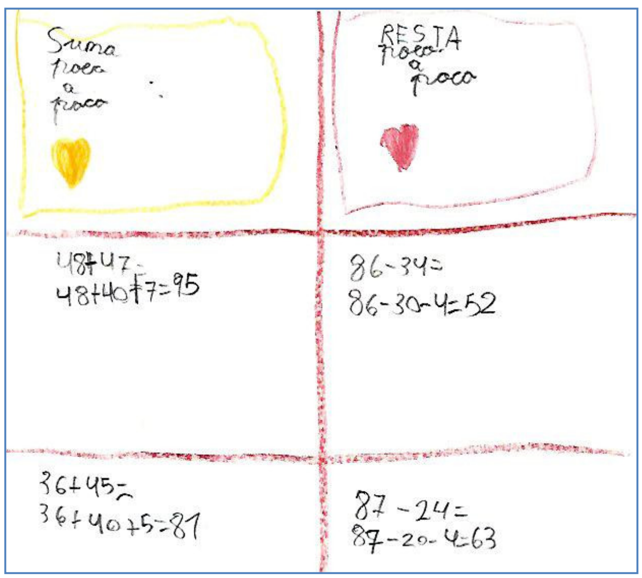


Figure 24. Add and subtract "little by little".

3.4. Skip Kangaroo!

This activity will use the character of the kangaroo. The kangaroo can jump forwards ("+" expression) or backwards ("−" expression) on the number line. Its jumps can be of different magnitude to change from one family to another and overcome an obstacle located in the exact ten. In Figure 25, for instance, the kangaroo is at number twenty-eight and it has to jump until it reaches some number in the family of the thirties.

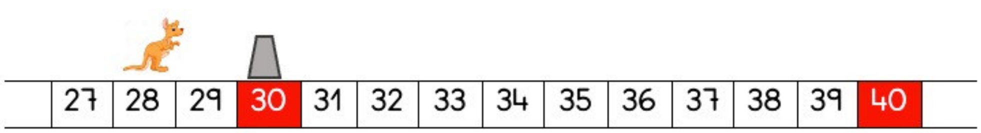


Figure 25. Kangaroo on the number line.

Once the kangaroo has been presented to the group of pupils, teachers should encourage an analysis of the situation from a broader perspective. They can ask questions such as: Which jumps should not go over the obstacle? What will be the minimum length of the jump? What will be the maximum length?

Students may answer: With a jump of three, it will reach number thirty-one.

The teacher will then encourage the students to express verbally all possible solutions and will accompany them with the corresponding written expression.

At the same time, it is important that the teacher checks on the number line if all the proposals are correct solutions.

Student: If it takes a jump of five, it reaches thirty-three

Teacher: How will we express this by writing what you have just said? (The teacher writes on the board at the same time they tell what kangaroo has made)

Teacher and Student: Begin at twenty-eight (28), jump forward five (+5) and reach thirty-three (=33).

On the board will be written:

$$28 + 5 = 33$$

Furthermore, then all possibilities are:

28 + 3 = 31	28 + 6 = 34	28 + 9 = 37
28 + 4 = 32	28 + 7 = 35	28 + 10 = 38
28 + 5 = 33	28 + 8 = 36	28 + 11 = 39

In this way, pupils can become aware that they have given full meaning to numbers and signs. They have translated each jump of the kangaroo into symbolic language. Going even deeper, the teacher can draw attention to quantities that stay the same and quantities that change, thus allowing them to intuitively grasp the concept of a variable.

Teacher: From what number has the kangaroo always jumped? What jumps has it made? With what jumps did it stay close to the start? With which did it get very far?...

With the implementation of this work in the classroom, so easy to understand and solve by the children, the way is being paved for the future understanding of expressions that involved the use of variables. In the case of the example above it would be $28 + x \leq 39$.

In the first two sessions with the kangaroo, the work should be focused at the sensorimotor and verbal level, favoring making decisions on the size and direction of the jump. With this activity, the teacher can gather much more information about the pupils' number sense than by simply doing calculations. In a third session, this activity can be performed on a written level. In support of the statement, an image of the tape with the corresponding numerical section can be included in the worksheet. Below, we show two cases with interesting results. Figure 26 shows that the student has identified a pattern so that the arrival number is always 61. The student controls the phenomena, something that is so present in relational thinking.

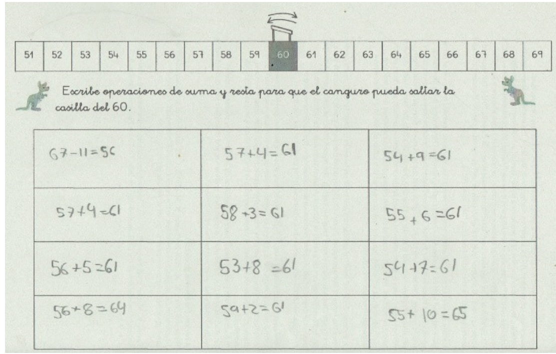


Figure 26. Kangaroo child activity 1: "Write additions and subtractions in such a way that the kangaroo overcomes number 60".

In Figure 27, the student ends, according to his own words, with "a jump from tip to tip":

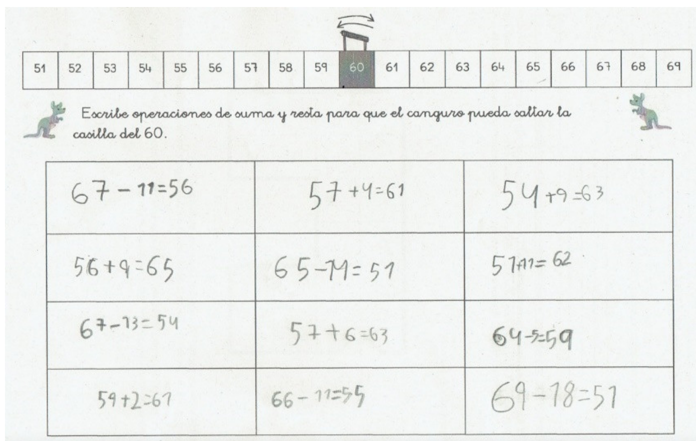


Figure 27. Kangaroo child activity 2: "Write additions and subtractions in such a way that the kangaroo overcomes number 60".

3.5. The frog Saltarina

In order to achieve agility and efficiency in thoughtful calculation, it is key to work on the convenient decomposition of numbers. It is not a simple skill for children because it requires observation and analysis of the whole operation, reflection and planning of the tactics to be used, and also so can a good mastery of numbers on a mental and written level. To work on the most appropriate decomposition for a given calculation, we make use of another character: the frog Saltarina. It is a frog that jumps back and forth across the number line (Figure 28). In addition, Saltarina is very fond of the color red and whenever it can, it makes an intermediate jump, stopping at the numbers with a red background.

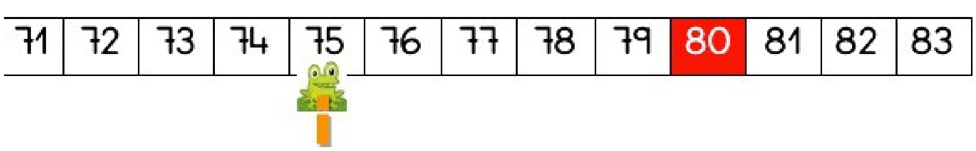


Figure 28. Saltarina is on 75.

Teacher: Today, Saltarina adds seven, where will it stop, and which number will it reach?

Student: It will jump five to land on eighty and later it will jump two.

Teacher: Well done! Let's check it!

The teacher performs the two movements with Saltarina, stopping at eighty and then reaching eighty-two.

Teacher: It has reached eighty-two! Let's write what has happened.

Students: Saltarina was on seventy-five (the teacher writes 75), first it has made a leap forward of five to reach eighty (she writes +5) and then it has made another leap of two (+2). It has reached eighty-two (Figure 29).

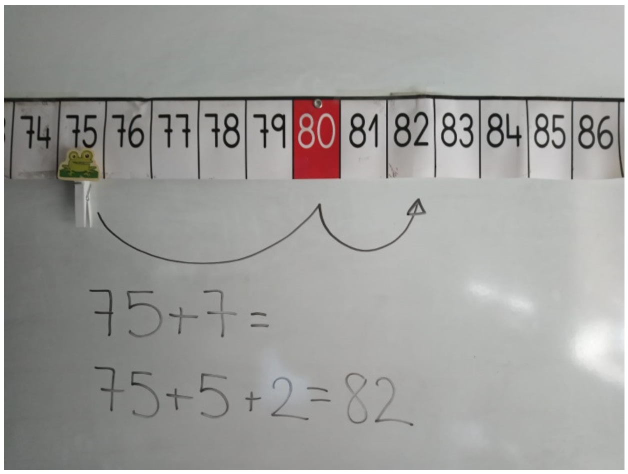


Figure 29. Saltarina's jump represented in a symbolic way.

Teacher: It has reached eighty-two!... What if it now adds nine?

Student: Now it has to jump eight to ninety and then one (Figure 30).

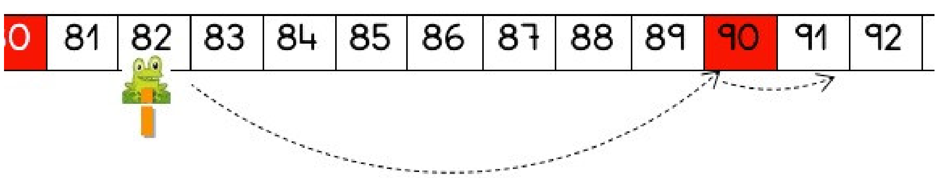


Figure 30. Saltarina is in 82 and jumps 9.

Teacher: And how do we narrate it with a calculation?

The teacher will write on the board the jump in two movements and the arrival number:

$$82 + 9 =$$
$$82 + 8 + 1 = 91$$

Below examples of two students' activities are shown: one in which Saltarina adds seven (Figure 31) and another which goes back eight (Figure 32). In both, the ENL has been used as a scheme for the representation of the movements performed:

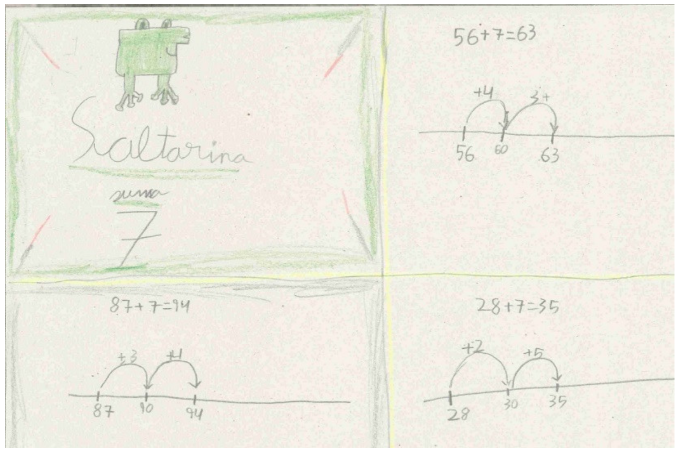


Figure 31. Saltarina's worksheet: it adds seven.

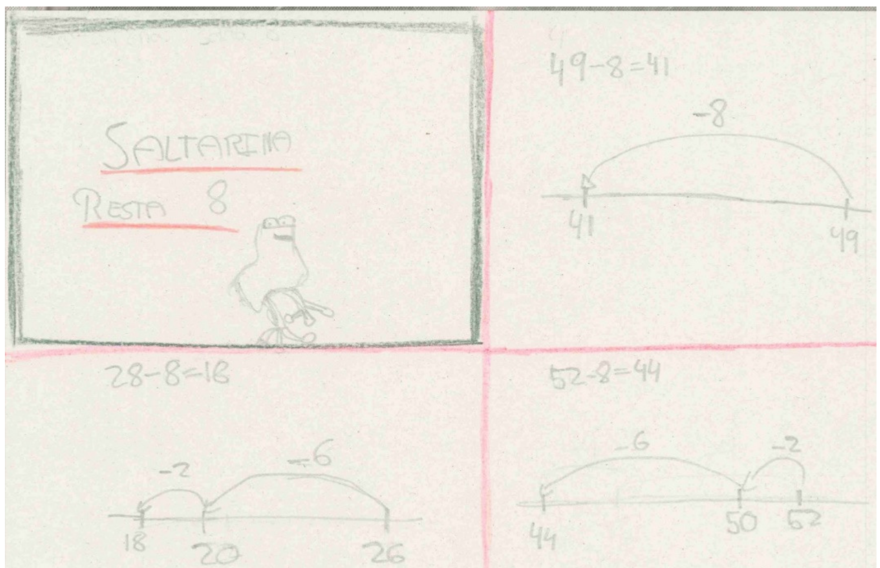


Figure 32. Saltarina's worksheet: it subtracts eight.

Next, we analyze some examples of tactical calculation, in which many of the strategies that have been worked on with the students are overturned, in which we have denominated a tactical calculation. Figure 33 shows the operation 41 − 35: the student rewrites explaining the operation plan that will follow, removing a unit, then three tens and finally four units. Then, she returns to the beginning and mentally performs very simple calculations: 41 − 1 = 40; 40 − 30 = 10; 10 − 4 = 6

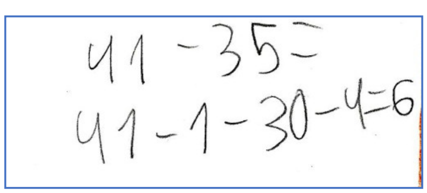


Figure 33. Work out 41 − 35.

In operation 53 + 29 (Figure 34), the student transforms the second adding into an equivalent expression more convenient for mental calculation, since he gets exact tens.

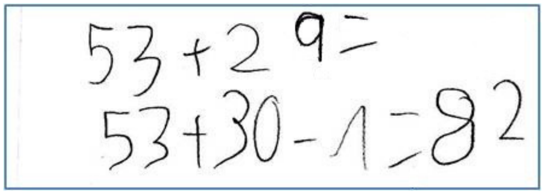


Figure 34. Work out 53 + 29.

In Figure 35 the operation 35 + 49 is shown, and to work it out, the student makes a transfer of one quantity to another, in this case he decides to round off 35 to forty:

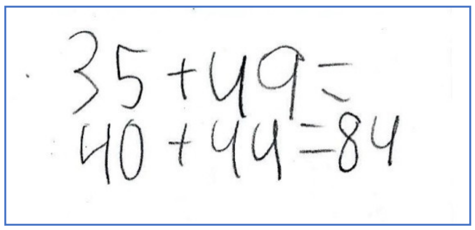


Figure 35. Work out 35 + 49.

If we show examples with larger numbers, we observe how they adapt their calculation strategies in a convenient way. In Figure 36 the student begins with a transfer tactic, passing the hundreds to the first quantity; with this, the expression has been simplified and leaves a better visualization of the section in which it operates. Then the second addition is decomposed by application of the "first row tactic", which is based on the generalization of the complements from ten to one hundred. The route of this action plan is easy for the brain:

$$649 + 60 = 709; 709 + 30 = 739; 739 + 2 = 741$$

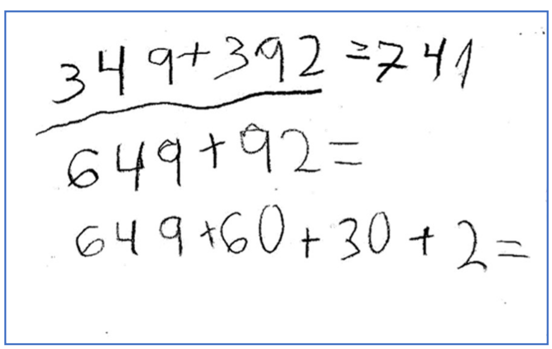


Figure 36. Work out 349 + 392.

The example in 948 − 757 (Figure 37) follows a procedure similar to the previous one. The student applies a preliminary tactic to simplify the operation by removing the hundreds. Then, he rewrites saying that he will remove 40 to return to the first row, then 10 and finally 7. Thus, the mental calculations are based on simple numerical skills:

$$248 - 40 = 208; 208 - 10 = 198; 198 - 7 = 191$$

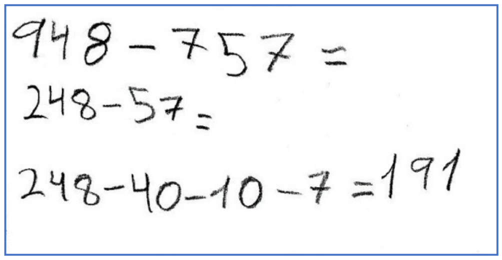


Figure 37. Work out 948 − 757.

Throughout the activities and examples shown, it becomes clear how the theoretical premises we started from and which were explained in the first part of the article are applied in practice. The debates generated and the students' productions when performing these and other activities represent a great source of information for observing the degree of acquisition and awareness of relational thinking, understanding and generalization of patterns.

4. Discussion

In a general way, for this work we started from widely verified assumptions, both from the own practical experience of the authors and through the investigations carried out by them and many other researchers within the researcher community of Mathematics Education which focuses on the development of arithmetic and algebraic thinking, such as:

- The importance of number sense development in the first years of mathematical learning, and the importance in this process of achieving significant learning of the decimal numbering system and a comprehensive and relational management of arithmetic operations and their properties, as suggested by [9,44]. All this is aligned with previous research that highlights that small changes in traditional arithmetic practice and learning environment can be key to conceptual understanding [7,45].
- The importance of algebra in secondary education, as a gateway to symbolic thinking in Mathematics, and the difficulties traditionally faced by both teachers and students of this educative level, in order to develop the competencies expected in this area.
- The steps that have been taken in recent years in research in mathematics education in relation to early algebra, in favor of an appropriate transition from concrete arithmetic to the symbolic language of algebra and its positive consequences in favor of abstract mathematical reasoning. Furthermore, in this sense, the relative abundance of specific examples, but the lack and the need to have reference methodological models that, with greater depth, make possible a real breakthrough in the context of practice in the classroom [13,14].
- Finally, the conviction that in the first years of school learning it is necessary to use manipulative resources for the construction of mathematical knowledge in general and the development of number sense in particular [10].

As a result, we set ourselves the objective of ordering the fundamental theoretical references necessary to equip the methodology, which is already being used in the first years of Primary Education, with the appropriate didactic resources to address those aspects of arithmetic learning that favor the development of the competences of early algebraic thinking.

This is how, in an initial phase of our work, based on a broad theoretical framework, we were able to "build" in a satisfactory way our unique "puzzle" designed to inspire and guide learning in the classroom towards a generalizable arithmetic in many aspects and in that sense "algebraic", from the use of elements that we consider fundamental, such as the visualization of properties and regularities through manipulation, the promotion of quantitative and relational thinking, the generalization of patterns and the flexible and "tactical" calculation.

The next step was to incorporate into our methodological references the appropriate didactic activities and resources for the development of the skills and abilities of early algebraic thinking that we have just referred to, thus forming a didactic guide, whose implementation has been carried out in a group classroom constituted by 25 students of the first year of Primary Education. To this end, and as an example, some records of the methodological intervention made have been collected and analyzed.

What it is more, in coherence with the methodology that was being developed, two important elements, in a collateral way, have arisen as key issues: on the one hand, the promotion of motivational aspects, both in the students, due to the characteristics of the activities that are developed, as in the course of the teaching activity. On the other hand, we also note important advances that are related to the level of representation and communication skills, both verbal and written, something that undoubtedly has special significance regarding the communicative competence that should be attributed to Mathematics, and which has already been acknowledged by other authors [8,9].

In conclusion, we find relevant a well-planned early intervention. From the earliest years of learning, such intervention, through the development of number sense, naturally enhances the transition from the specific to the generalizable and, beyond that, from the narration of purely arithmetical facts to the symbolic language more characteristic of algebraic thinking, as already suggested by [7,8,46]. This kind of methodological approach must last throughout the educational stage of Primary Education.

However, we understand this work as a first approach. Its evidence should be tested empirically through wider research. With the objective of achieving an integral perspective, it should have a double qualitative and descriptive-quantitative component to observe aspects, both motivational and related to the impact on learning in the development of mathematical competence related to algebraic thinking. The ideas of [47] might be followed in such a research, which will be addressed in forthcoming papers.

Author Contributions: Conceptualization, N.A.-P., E.F.-A. and M.T.G.-P.; Data curation, N.A.-P., E.F.-A., M.T.G.-P. and J.M.-G.; Formal analysis, N.A.-P., E.F.-A., M.T.G.-P. and J.M.-G.; Funding acquisition, N.A.-P. and E.F.-A.; Methodology, N.A.-P. and J.M.-G.; Supervision, N.A.-P. and M.T.G.-P.; Writing—original draft, N.A.-P., E.F.-A. and M.T.G.-P.; Writing—review and editing, E.F.-A. and J.M.-G. All authors have read and agreed to the published version of the manuscript.

Funding: Part of this research was funded by the University of Cordoba, in the framework of the project 2020-4-4003—Methodological innovation in the area of school arithmetic in the first cycle of Primary Education.

Acknowledgments: The authors are grateful to the management team of the Al-Andalus school for facilitating data collection and observation of classroom practices.

Conflicts of Interest: The authors declare no conflict of interest.

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